Pricing of Variance Swaps in a Mean-Reverting Gaussian Volatility Model

Study carried out by the Quantitative Practice
Special thanks to Pierre-Edouard THIERY

awalee

AVRIL 2022

Summary

Introduction

Trading volatility is not the newest idea in finance, even though it is not something which is as straightforward as the trading of other asset since volatility is not a tradable asset in itself. It is only a quantity which is related to another tradable asset. However, due to the strong interest in volatility, volatility products have emerged over the last few decades, the most popular one being the variance swap.

The pricing of variance swaps soon gained much mathematical interest. There are two types of valuation methods: analytical and numerical. Among the analytical methods, the pioneer work by Carr, Madan and Demeterfi and al.[1] [2] has played an important role: it established that it is possible to replicate the cumulated, continus, realized variance over a given interval by a static positions in a continuum of options, plus dynamic positions in futures. Those results, albeit of the utmost theoretical importance, face several challenges, the most severe ones being the fact the there is no such thing as a continuum of options for all strikes, and that the realized volatility is defined through an integral [3]. However it is worth noticing that this replicating approaches is model-free.

Analytical approaches have also been studied in a model-dependent framework, most of the models relying of a stochastic volatility [4]. However, once again, the realized volatility is apprehended with a purely mathematical point of view, as a quantity which is defined continuously through an integral, instead of choosing a discrete definition, as it is the case on financial markets.

To circumvent this issue, numerical methods have been developed [5] [6]. Nonetheless, those methods do not incorporate stochastic volatilities and all their characteristics.

Therefore much attention has been paid recently on the question of pricing variance swaps defined discretely with a stochastic volatility model [7] [8] [9], [10], for instance a Heston model [11]. In our paper, we choose to work with a mean-reverting Gaussian volatility model [12]. Such a model has already been used for the pricing a variance swap defined discretely thanks to real returns: some interesting techniques have been introduced, such dimension reduction and the use of the Fourier transform, to solve this pricing problem. In our work, we adapt those techniques to resolve a slightly different, but harder, problem: the pricing of variance swap defined discretely with log-returns. Using log-returns to define a variance swap is indeed a widespread alternative to real returns on financial markets, but such a choice adds a layer of mathematical complexity when it comes to pricing.

1 The Pricing Framework

1.1 The Mean-Reverting Gaussian Volatility Model

Before considering volatility products such as variance swaps, we must first define a model for the asset price. We choose a mean-reverting Gaussian model [12], meaning that the volatility of the asset is assumed to follow an Ornstein-Uhlenbeck process. If we denote (S_t) the price of the asset, we can write:

$$\begin{cases} dS_t = \mu S_t dt + \nu_t S_t dB_t^S \\ d\nu_t = \tilde{\kappa}(\tilde{\theta} - \nu_t)dt + \sigma dB_t^\nu \end{cases}$$

where B^S and B^ν are two Brownian motions under the historical probability $\mathbb{P}$, such that $\left\langle dB_t^S, dB_t^\nu \right\rangle = \rho dt$, μ is the expected return of the asset, $\tilde{\theta}$ the long-term mean of its volatility, κ the speed of the mean-reverting behavior of the asset volatility, σ the volatility of volatility.

If we assume that the market is arbitrage-free, it means that there exists a risk-neutral probability, which we denote $\mathbb{Q}$. Under such a probability, the dynamics of the asset can be rewritten with the risk-free rate r:

$$\begin{cases} dS_t = rS_t dt + \nu_t S_t dW_t^S \\ d\nu_t = \kappa(\theta - \nu_t)dt + \sigma dW_t^\nu \end{cases}$$

where W^S and W_ν are two Brownian motions under $\mathbb{Q}$, exhibiting the same constant correlation ρ.

1.2 The Payoff of Variance Swaps

If we consider a time period $[0, T]$, a variance swap on a given underlying S over such a period is a contract between two parties: at maturity, the first one pays the second one the realized variance of the asset over $[0, T]$, while the second one pays the first one a predetermined amount, known as the variance strike of the contract. So, if we denote $\mathcal{N}$ the notional of the contract, $\sigma_R^2(0, T)$ the annualized realized variance and K_v the variance strike, the payoff at maturity of the variance swap is equal to:

$$V_T = \left(\sigma_R^2(0, T) - K_v\right) \mathcal{N}$$

To compute the realized variance, it is first necessary to consider a discretization of the time period $[0, T]$ with N points: $N\Delta t = T$ and $t_i = i\Delta_t$. Then, in real life, there are two ways of defining the realized variance: this may be done either with actual returns, or with log-returns. With actual returns, the annualized realized variance is given by:

$$\sigma_R^2(0, T, N) = \frac{A}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2$$

With log-returns, the annualized variance is equal to:

$$\sigma_R^2(0, T, N) = \frac{A}{N} \sum_{i=1}^{N} \ln^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right)$$

Two remarks are worth noticing. First, the annualization depends on the sampling frequency. For instance, if Δt is equal to one business day, since there are typically 252 business days in a year, $A = 252$. With a weekly sampling, A would be equal to 52, and so on and so forth. Second, we do not subtract the mean of the returns in the above formulas: considering that such a mean is equal to 0 is a common assumption when it comes to variance swaps in real life. Thanks to this assumption, variance swaps exhibit an additive property: it is possible to construct a variance swap over a longer period of time by considering two variance swaps on shorter periods of time.

According to the theory of pricing, the value V_t of the variance swap at date $t \leq T$ is given by the following expectation under the risk-neutral probability $\mathbb{Q}$:

$$V_t = \mathbb{E}_t^{\mathbb{Q}} \left[e^{-r(T-t)} \left(\sigma_R^2(0,T) - K_v \right) \mathcal{N} \right]$$

There is usually no cost for any party to enter a variance swap: there is no exchange of money at the inception of the contract. This means that the value V_0 should be equal to 0. This enables us to determine the fair value of the variance strike K_v:

$$K_v^{fair} = \mathbb{E}_0^{\mathbb{Q}} \left[\sigma_R^2(0,T) \right]$$

This fair variance strike is important since it ensures that the variance swap is fair, meaning that nothing needs to be paid at the inception of the contract. If the strike is not equal to the fair variance strike, one of the parties must pay an amount of money to the other one so as to make sure that the contract is fair. Typically, the fair variance strike is used when entering a variance swap. Therefore the valuation of a variance swap amounts to the computations of the fair variance strike.

As mentioned in the introduction, we focus in our work on the log-return case, so the fair variance strike is equal to:

$$\mathbb{E}_0^{\mathbb{Q}} \left[\sigma_R^2(0,T) \right] = \frac{A}{N} \sum_{i=1}^{N} \mathbb{E}_0^{\mathbb{Q}} \left[\ln^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \right]$$

Thus the valuation of a variance can be seen as the computations of the N expectations:

$$\mathbb{E}_0^{\mathbb{Q}} \left[\ln^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \right]$$

where S follows a mean-reverting Gaussian volatility model. When $i = 1$, the situation is slightly simpler, since S_0 is known; however, when $i > 0$, the expectations is based on two random variables, S_{t_i} and $S_{t_{i-1}}$.

2 How To Mathematically Handle The Expectation?

2.1 From One To Two Problems

As of now, we work for a given index i. For the sake of readability, the index i will be omitted in most of the notations; if the index i is present, it is only to remind the reader that we work for a fixed index i.

In order to mathematically compute the expectation $\mathbb{E}_0^{\mathbb{Q}} \left[\ln^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \right]$, we start by introducing a new variable, denoted D (instead of D^i):

$$D_t = \int_0^t \delta(t_{i-1} - s) S_s ds$$

where $\delta(x)$ denotes the Dirac function in x. A simpler way of rewriting D_t is:

$$D_t = \begin{cases} S_{t_{i-1}} & when\ t_{i-1} \leq t \leq t_i \\ 0 & when\ 0 \leq t < t_{i-1} \end{cases}$$

The advantage of the integral form of D is just that it gives the differential form: $dD_t = \delta(t_{i-1} - t) S_t dt$.

Let us now assume that we consider a derivative whose payoff at maturity t_i is $\ln^2 \left(\frac{S_{t_i}}{D_{t_i}} \right)$. By definition, this payoff is equal to $\ln^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right)$. We denote U_t the value at t of such a contract. If we write:

$$U_t = U(S_t, \nu_t, D_t, t)$$

where U is a four-variable function, with variables S, ν, D and t, we have the following result:

Theorem 1: Partial Derivatives Equation
The function U is such that:

$$\partial_t U + \frac{1}{2} S^2 \nu^2 \partial_{SS} U + \frac{1}{2} \sigma^2 \partial_{\nu\nu} U + \rho S \nu \sigma \partial_{S\nu} U$$
$$+ r S \partial_S U + \kappa(\theta - \nu) \partial_\nu U + \delta(t_{i-1} - t) S \partial_D U - rU = 0$$

with the terminal condition:

$$U(S, \nu, D, t_i) = \ln^2 \left(\frac{S}{D} \right)$$

Albeit impressive, the PDE is not very difficult to establish. To do so, we start by applying Ito's formula to the process $U_t = U(S_t, \nu_t, D_t, t)$. This gives us:

$$dU_t = \left[\partial_t U + \frac{1}{2} S_t^2 \nu_t^2 \partial_{SS} U + \frac{1}{2} \sigma^2 \partial_{\nu\nu} U + S_t \nu_t \rho \sigma \partial_{S\nu} U \right.$$
$$\left. r S_t \partial_S U + \kappa(\theta - \nu_t) \partial_\nu U + \delta(t_{i-1} - t) S_t \partial_D U \right] dt$$
$$+ S_t \nu_t \partial_S U dW_t^S + \sigma \partial_\nu U dW_t^\nu$$

To establish the PDE, we then assume we try to reproduce the payoff at maturity of our contract using a self-financing portfolio P. Since we have two sources of risk in our model, W^S and W^ν, we consider two risky assets, each one being associated to one source of risk, plus a risk-free asset S^0. The first risky asset is the most natural one in our situation: S. Since there is a second source of risk, we assume that there exists a tradable asset C, whose source of risk is precisely W^ν. We denote it C. By definition, under the risk-neutral probability, C has the following dynamics:

$$dC_t = rC_t dt + \gamma_t C_t dW_t^\nu$$

where γ_t is the volatility of C.

Our portfolio can then be written:

$$P_t = \delta_t S_t + w_t C_t + \delta_t^0 S_t^0$$

Using the self-financing assumption, we have:

$$dP_t = \delta_t dS_t + w_t dC_t + \delta_t^0 dS_t^0$$

$$= dt\left[r(U_t - \delta_t S_t - w_t C_t) + \delta_t S_t r + w_t C_t r\right]$$

$$+\delta_t S_t \nu_t dW_t^S + w_t C_t \gamma_t dW_t^{\nu_t}$$

Now, since our self-financing portfolio P_t aims at reproducing the terminal payoff, we have $U_t = P_t$: the price of a contract is the cost of its hedge. Identifying the differential terms, we get the PDE of Theorem 1.

Then, using Feynman-Kac formula, we know that the premium to be paid for this derivative at $t = 0$ is worth:

$$U(S_0, \nu_0, D_0, 0) = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-rt_i} U(S_{t_i}, \nu_{t_i}, D_{t_i}, t_i)\right]$$

So

$$\mathbb{E}_0^{\mathbb{Q}}\left[\ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right] = e^{rt_i} U(S_0, \nu_0, D_0, 0)$$

The gist is to find the desired expectation by finding the solution of the aforementioned PDE. To do so, we are going to work, first on t_{i-1}, t_i, then on $[0, t_{i-1}[$. On both intervals,the PDE can be rewritten:

$$\partial_t U + \frac{1}{2}S^2\nu^2\partial_{SS}U + \frac{1}{2}\sigma^2\partial_{\nu\nu}U + \rho S\nu\sigma\partial_{S\nu}U$$

$$+rS\partial_S U + \kappa(\theta - \nu)\partial_\nu U - rU = 0$$

First we solve the PDE on the interval t_{i-1}, t_i, using the terminal condition of theorem 1: $U(S, \nu, D, t_i) = \ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$. Then, we solve the same PDE on the interval $[0, t_{i-1}[$, with a terminal condition induced from the solution of the previous problem.

2.2 Start With The End: The First Problem

Let us assume we consider a derivative of maturity T, on the underlying S following the dynamics stated at the beginning of this paper, and whose payoff at T is $H(S)$. We have the following theorem:

Theorem 2: General Solution
If the replicating portfolio (U_t) is written $U_t = U(S_t, \nu_t, t)$, meaning that the function U is the solution of:

$$\partial_t U + \frac{1}{2}S^2\nu^2\partial_{SS}U + \frac{1}{2}\sigma^2\partial_{\nu\nu}U + \rho S\nu\sigma\partial_{S\nu}U$$

$$+rS\partial_S U + \kappa(\theta - \nu)\partial_\nu U - rU = 0$$

with the terminal condition $U(S, \nu, T) = H(S)$, then U can be written:

$$U(S, \nu, t) = V(\ln(S), \nu, T - t)$$

where

$$V(x, \nu, \tau) = \mathscr{F}^{-1}\left[w \mapsto e^{C(w,\tau)+D(W,\tau)\nu+E(w,\tau)\nu^2}\mathscr{F}\left\{x \mapsto H(e^x)\right\}(w)\right]$$

$\mathscr{F}$ denotes the Fourier Transform operator with the convention

$$\mathscr{F}(f)(w) = \int_{-\infty}^{\infty} f(x)e^{-iwx}dx$$

and the functions C, D and E are:

$$\begin{cases} E(w,\tau) = \dfrac{c_1(w)(-1 + e^{2\beta(w)\tau})}{\alpha(w)(-1 + e^{2\beta(w)\tau}) + \beta(w)(1 + e^{2\beta(w)\tau})} \\ D(w,\tau) = -\dfrac{2c_1(w)(-1 + e^{\beta(w)\tau})\theta\kappa}{\beta(w)\left[\alpha(w)(-1 + e^{2\beta(w)\tau}) + \beta(w)(1 + e^{2\beta(w)\tau})\right]} \\ C(w,\tau) = \dfrac{1}{2}\ln\left(\dfrac{\alpha(w)(-1 + e^{2\beta(w)\tau}) + \beta(w)(1 + e^{2\beta(w)\tau})}{2\beta(w)}\right) \end{cases}$$

$$+ \tau\left(b(w) - \frac{\beta(w) + \alpha(w)}{2} - \frac{c_1(w)\theta^2\kappa^2}{\beta(w)^2}\right)$$

$$+\frac{\theta^2\kappa^2 c_1(w)\left[2\alpha(w)(-1 + e^{\beta(w)\tau})^2 + \beta(w)(-1 + e^{2\beta(w)\tau})\right]}{\beta(w)^3\left[\alpha(w)(-1 + e^{2\beta(w)\tau}) + \beta(w)(1 + e^{2\beta(w)\tau})\right]}$$

with $\alpha(w) = \kappa - i\rho\sigma w$, $c_1(w) = \frac{w^2+iw}{2}$, $\beta(w) = \sqrt{\alpha(w)^2 + 2c_1(w)\sigma^2}$, and $b(w) = (iw - 1)r$.

This theorem may seem rather fastidious, however, beyond pure technical computations, its demonstration relies on a few simple ideas.

First we carry out two changes a variable, by setting $x = \ln(S) \Leftrightarrow S = e^x$, and $\tau = T - t$. We write $V(x, \nu, \tau) = U(e^x, \nu, T - t)$. The PDE on U turns into a PDE on V:

$$\partial_\tau V = \frac{1}{2}\nu^2\partial_{xx}V + \rho\sigma\nu\partial_{x\nu}V + \frac{1}{2}\sigma^2\partial_{\nu\nu}V$$

$$+\left[r - \frac{1}{2}\nu^2\right]\partial_x V + \kappa(\theta - \nu)\partial_\nu V - rV$$

with the initial condition $V(x, \nu, 0) = U(e^x, \nu, T) = H(e^x)$.

We then apply the Fourier transform on the functions whose variable is x to turn them into new functions with variable w. To do so, we use the following result on Fourier transform:

$$\mathscr{F}(f^{(n)})(w) = (iw)^n\mathscr{F}(f)(w)$$

We denote $\tilde{V}$ the Fourier transform of V. The PDE on V turns into a PDE on $\tilde{V}$:

$$\partial_\tau\tilde{V} = \frac{1}{2}\sigma^2\partial_{\nu^2}\tilde{V} + \tilde{V}\left[-\frac{1}{2}\nu^2(w^2 + iw) + riw - r\right]$$

$$+\partial_\nu\tilde{V}\left[\kappa\theta - \kappa\nu + \nu\rho\sigma iw\right]$$

with initial condition: $\tilde{V}(w, \nu, 0) = \mathscr{F}(x \mapsto H(e^x))(w)$.

We now assume that there exists a solution V of the following form:

$$\tilde{V}(w, \nu, \tau) = e^{C(w,\tau)+D(w,\tau)\nu+E(w,\tau)\nu^2}\tilde{V}(w, \nu, 0)$$

Then, it is possible to show that the functions E, D and C are necessarily the solutions of a Riccati system

$$\begin{cases} \partial_\tau E = 2\sigma^2E^2 - 2\alpha(w)E - c - 1(w) \\ \partial_\tau D = 2\sigma^2DE - \alpha(w)D + 2E\kappa\theta \\ \partial_\tau C = \dfrac{1}{2}\sigma^2(2E + D^2) + \kappa\theta D + b(w) \end{cases}$$

Such a system can be resolved using common techniques regarding Riccati systems: the solutions is given by the three expressions displayed in Theorem 2.

So, if $\tilde{V}$ follows the above-mentioned form, necessarily C, D and E are the expressions of Theorem 2. Reciprocally, if we consider such a function $\tilde{V}$, it is straightforward, albeit a bit long, to see that it verifies the desired PDE.

This gives us that:

$$V(x,\nu,\tau) = \mathscr{F}^{-1}\left[w \mapsto e^{C(w,\tau)+D(w,\tau)\nu+E(w,\tau)\nu^2}\mathscr{F}\left(x \mapsto H(e^x)\right)(w)\right]$$

and so the desired result, using $U(S,\nu,t) = V(\ln(S),\nu,T-t)$.

After this general theorem, let us go back to our pricing issue. In order to find the solution of our first PDE on t_{i-1}, t_i, we use the general theorem as well as another common result regarding the Fourier transform:

$$\mathscr{F}\left[x \mapsto x^n\right](w) = 2\pi i^n \delta^{(n)}(w)$$

where $\delta^{(n)}$ is the distribution such that:

$$\int_{-\infty}^{\infty} \delta^{(n)}\Phi(w)dw = (-1)^n \Phi^{(n)}(0)$$

In our case, the function H (terminal condition of the general theorem) is simply equal to:

$$H(S) = \ln^2\left(\frac{S}{D}\right)$$

So $H(e^x) = (x-\ln(D))^2 = x^2 - 2\ln(D)x + \ln^2(D)$. If we write λ is Fourier transform:

$$\lambda(w) = 2\pi\left[-\delta^{(2)}(w) - 2i\ln(D)\delta^{(1)}(w) + \ln^2(D)\delta(w)\right]$$

Since the inverse Fourier transform is given by:

$$\mathscr{F}^{-1}(w \mapsto g(w)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(w)e^{iwx}dw$$

This means that

$$V(x,\nu,\tau) = \mathscr{F}^{-1}\left[w \mapsto e^{C(w,\tau)+D(w,\tau)\nu+E(w,\tau)\nu^2}\lambda(w)\right]$$

$$= \int_{-\infty}^{\infty} e^{C(w,\tau)+D(w,\tau)\nu+E(w,\tau)\nu^2+iwx}$$

$$\left\{-\delta^{(2)}(w) - 2i\ln(D)\delta^{(1)}(w) + \ln^2(D)\delta(w)\right\}dw$$

Then V can be written:

$$V(x,\nu,\tau) = \ln^2(D)f(0;x,\nu,\tau) + 2i\ln(D)f'(0;x,\nu,\tau) - f''(0;x,\nu,\tau).$$

where

$$f(w;x,\nu,\tau) = e^{C(w,\tau)+D(w,\tau)\nu+E(w,\tau)\nu^2+iwx}$$

In this expression, we clearly separate w and the other variables x, ν and τ, which can be seen as mere parameters. The derivative is now taken regarding the true variable, i.e. w.

This gives us the solution of the first PDE, on the time period t_{i-1}, t_i:

$$U(S,\nu,D,t) = \ln^2(D)f(0;\ln(S),\nu,t_i - t)$$

$$+2i\ln(D)f'(0;\ln(S),\nu,t_i-t) - f''(0;\ln(S),\nu,t_i-t)$$

So the price of our derivative contract between t_{i-1} and t_i is equal to $U(S_t,\nu_t,D_t,t)$.

2.3 The Second Problem

We now have to solve the same PDE, but on the interval $[0, t_{i-1}[$. The terminal condition is at t_{i-1} and depend on the solution of the PDE on t_{i-1}, t_i.

By continuity of the price at t_{i-1}, we know that the price at t_{i-1} is equal to $U(S_{t_{i-1}},\nu_{t_{i-1}},D_{t_{i-1}},t_{i-1})$, where U is the solution of the first problem.

In order to find the initial price of the derivative paying $\ln^2\left(\frac{S_{t_i}}{D_{t_i}}\right)$, we are going to consider another pricing problem, with maturity t_{i-1}. However, to precisely define this new pricing problem, first we must refine the expression of $U(S_{t_{i-1}},\nu_{t_{i-1}},D_{t_{i-1}},t_{i-1})$, which is going to provide us with the new terminal payoff.

Mathematically, we compute the limit, when $t > t_{i-1}$, $\lim_{t\to t_{i-1}} U(S,\nu,D,t)$. Indeed, from a mathematical point of view, the function U is not define for $t = t_{i-1}$.

Theorem 3: New Terminal Payoff
The price of the derivative paying $\ln^2\left(\frac{S_{t_i}}{D_{t_i}}\right)$ *at maturity* t_i *can be written at* t_{i-1}*:*

$$U(S_{t_{i-1}},\nu_{t_{i-1}},D_{t_{i-1}},t_{i-1}) = e^{r\Delta t}g(\nu_{t_{i-1}})$$

where $\Delta t = t_i - t_{i-1}$ *and* g *a known polynomial function.*

Once again, this theorem may seem fastidious to establish, but the proof is actually rather simple. Indeed, we have just seen that:

$$U(S,\nu,D,t) = \ln^2(D)f(0;\ln(S),\nu,t_i-t)$$

$$+2i\ln(D)f'(0;\ln(S),\nu,t_i-t) - f''(0;\ln(S),\nu,t_i-t)$$

First

$$f(0;\ln(S),\nu,t_i-t) \xrightarrow{t\to t_{i-1}} e^{C(0,\Delta t)+D(0,\Delta t)\nu+E(0,\Delta t)\nu^2+0}$$

But it is straightforward to verify that: $E(0,\Delta t) = 0$, $D(0,\Delta t) = 0$ and $C(0,\Delta t) = -r\Delta t$. So we simply find:

$$f(0;\ln(S),\nu,t_i-t) \xrightarrow{t\to t_{i-1}} e^{-r\Delta t}$$

This result is very helpful since it ensures that the calculations are quite simple. If we focus on $f'(w;\ln(S),\nu,t_i-t)$, we have:

$$f'(w;\ln(S),\nu,t_i-t) = f(w;\ln(S),\nu,t_i-t)$$

$$\left\{\partial_w C(w,t_i-t) + \partial_w D(w,t_i-t)\nu + \partial_w E(w,t_i-t)\nu^2 + i\ln(S)\right\}$$

So when $t \to t_{i-1}$ and $w = 0$, we find that the limit of the first derivative of f (whose sole variable is w) is equal to:

$$e^{-\Delta t}\left\{\partial_w C(0,\Delta t) + \partial_w D(0,\Delta t)\nu + \partial_w E(0,\Delta t)\nu^2 + i\ln(S)\right\}$$

The same goes for the second derivative: it is just that the expressions are slightly longer. We do not display here all the calculations insofar as they are similar to what was done for the first derivative.

At the end, we find that the expression of the limit of $U(S, \nu, D, t)$ when $t \to t_{i-1}$ is equal to:

$$e^{-\Delta t}\left(g(\nu) + \ln^2\left(\frac{S}{D}\right)\right)$$

where g is the following function:

$$g(\nu) = -\partial_{ww}C(0,\Delta t) - \partial_{ww}D(0,\Delta t)\nu - \partial_{ww}E(0,\Delta t)\nu^2$$

$$-\left(\partial_w C(0,\Delta t)\right)^2 - \left(\partial_w D(0,\Delta t)\right)^2\nu^2 - \left(\partial_w E(0,\Delta t)\right)^2\nu^4$$

$$-2\partial_w C(0,\Delta t)\partial_w D(0,\Delta t)\nu - 2\partial_w C(0,\Delta t)\partial_w E(0,\Delta t)\nu^2$$

$$-2\partial_w E(0,\Delta t)\partial_w D(0,\Delta t)\nu^3$$

Beyond the length of the expression, it is important to notice that g is merely a polynomial function.

By replacing S by $S_{t_{i-1}}$, ν by ν_{ti-1} and D by $D_{t_{i-1}}$, we find the price at t_{i-1} of the derivative paying $\ln^2\left(\frac{S_{t_i}}{D_{t_i}}\right)$ at maturity t_i:

$$e^{-r\Delta t}g(\nu_{t_{i-1}})$$

because $\ln^2\left(\frac{S_{t_{i-1}}}{D_{t_{i-1}}}\right) = \ln^2(1) = 0$.

This price can now be used as the terminal payoff of a new pricing problem on the interval $[0, t_{i-1}]$.

Theorem 4: General Solution

We consider a derivative on the underlying S following a mean-reverting Gaussian volatility model: its maturity is denoted T and its terminal payoff is $F(\nu_T)$. If the replicating portfolio is written $P_t = P(S_t, \nu_t, t)$, meaning that the function P is the solution of the PDE:

$$\partial_t P + \frac{1}{2}S^2\nu^2\partial_{SS}P + \frac{1}{2}\sigma^2\partial_{\nu\nu}P + \rho S\nu\sigma\partial_{S\nu}P$$

$$+rS\partial_S P + \kappa(\theta - \nu)\partial_\nu P - rP = 0$$

with terminal condition $P(S, \nu, T) = F(\nu)$, then P can be written:

$$P(S, \nu, t) = \int_{-\infty}^{\infty} e^{-r(T-t)}F(z)d(z; t, T, \nu)dz$$

where d_t is the following Gaussian density:

$$d(z; t, T, \nu) = \frac{1}{\sqrt{2\pi}\bar{\sigma}(t, T, \nu)}e^{-\frac{(z-\bar{\mu}(t,T,\nu))^2}{2\bar{\sigma}(t,T,\nu)}}$$

with $\bar{\mu}(t, T, \nu) = e^{-\kappa(T-t)}\nu + \theta(1 - e^{-\kappa(T-t)})$ and $\bar{\sigma}(t, T, \nu) = \frac{\sigma^2}{2\kappa}(1 - e^{2\kappa(T-t)})$.

The proof relies on the Feynman-Kac formula: thanks to it, we know that

$$P(S, \nu, t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}F(\nu_T)\,|S_t = S, \nu_t = \nu\right]$$

The only random variable in this expectation is ν_T. To determine its law, we apply Ito's formula to the process $e^{\kappa t\nu_t}$:

$$d(e^{\kappa t}\nu_t) = \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}dW_t^\nu$$

Then, after integrating the above equation between t and T:

$$\nu_T = e^{-\kappa(T-t)}\nu_t + \theta(1 - e^{-\kappa(T-t)}) + \int_0^t \sigma e^{\kappa(s-T)}dW_s^\nu$$

The function in the stochastic integral is deterministic, meaning that the integral merely follows a normal law. ν_T is also a Gaussian variable, whose mean and variance are known:

$$\nu_T \equiv \mathcal{N}\left(e^{-\kappa(T-t)}\nu_t + \theta(1 - e^{-\kappa(T-t)}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(T-t)})\right)$$

We now apply Theorem 4 to our second pricing problem, on the interval $[0, t_{i-1}]$: maturity is t_{i-1} and the payoff function F is worth the polynomial function $e^{-r\Delta t}g$.

By construction, the initial value of this second problem is also the initial value of our overall pricing question (maturity t_i, payoff $\ln^2\left(\frac{S_{t_i}}{D_{t_i}}\right)$).

Theorem 5

Based on the above reasoning, we have:

$$\mathbb{E}_0^{\mathbb{Q}}\left[\ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right] = e^{rt_i}\int_{-\infty}^{\infty} e^{-rt_{i-1}}e^{-r\Delta t}g(z)d(z; 0, t_{i-1}, \nu_0)dz$$

$$= \int_{-\infty}^{\infty} g(z)d(z; 0, t_{i-1}, \nu_0)dz = \mathbb{E}^{\mathbb{Q}}\left[g(\nu_{t_{i-1}})\,|S_0, \nu_0\right]$$

The final calculations are fairly easy. Indeed, g is a polynomial function with degree 4, and its coefficients are known. For the sake of readibilty, we denote:

$$g(z) = \sum_{p=0}^{4} a_p z^p$$

Therefore:

$$\mathbb{E}_0^{\mathbb{Q}}\left[\ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right] = \sum_{p=0}^{4} a_p\mathbb{E}_0^{\mathbb{Q}}\left[\nu_{t_{i-1}}^p\right]$$

We only need to use the four first moments of Gaussian random variable:

$$\mathbb{E}_0^{\mathbb{Q}}\left[\nu_{t_{i-1}}\right] = \bar{\mu}(0, t_{i-1}, \nu_0)$$

$$\mathbb{E}_0^{\mathbb{Q}}\left[\nu_{t_{i-1}}^2\right] = \bar{\sigma}(0, t_{i-1}, \nu_0)^2 + \bar{\mu}(0, t_{i-1}, \nu_0)^2$$

$$\mathbb{E}_0^{\mathbb{Q}}\left[\nu_{t_{i-1}}^3\right] = 3\bar{\mu}(0, t_{i-1}, \nu_0)\bar{\sigma}(0, t_{i-1}, \nu_0)^3 + \bar{\mu}(0, t_{i-1}, \nu_0)^3$$

$$\mathbb{E}_0^{\mathbb{Q}}\left[\nu_{t_{i-1}}^4\right] = 3\bar{\sigma}(0, t_{i-1}, \nu_0)^4 + 6\bar{\sigma}(0, t_{i-1}, \nu_0)^2\bar{\mu}(0, t_{i-1}, \nu_0)^2$$

$$+\bar{\mu}(0, t_{i-1}, \nu_0)^4$$

So we see that, using the expression for the mean and the variance as stated in Theorem 4, and the expression of coefficients a_p, that

$$\mathbb{E}_0^{\mathbb{Q}}\left[\ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]$$

can be rewritten as a polynomial function of the initial volatility value ν_0. This function also depends on t_{i-1}. Furthermore, it is specific to the problem we have consider for index i. To take into account all those dependencies, we denote $h_i(t_{i-1}, \nu_0)$:

$$\mathbb{E}_0^{\mathbb{Q}}\left[\ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right] = h_i(t_{i-1}, \nu_0)$$

3 When All The Pieces Are Put Together

We saw in the first part of our paper that the valuation of a variance swap amounts to:

$$\mathbb{E}_0^{\mathbb{Q}}\left[\sigma_R^2(0,T)\right] = \frac{A}{N}\sum_{i=1}^{N}\mathbb{E}_0^{\mathbb{Q}}\left[\ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]$$

To compute such a quantity, we consider the N expectations: for $i = 1..N$, it is possible to prove that

$$\mathbb{E}_0^{\mathbb{Q}}\left[\ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right] = h_i(t_{i-1}, \nu_0)$$

where h_i is a known function, which is in particular polynomial with respect to its second variable.

This result was established in the second part of our work by considering two consecutive theoretical pricing problems, one on t_{i-1}, t_i, the second on $[0, t_{i-1}[$.

So, it is possible to establish an analytical formula for the fair variance strike of a variance swap:

$$K_v^{fair} = \mathbb{E}_0^{\mathbb{Q}}\left[\sigma_R^2(0,T)\right] = \frac{A}{N}\sum_{i=1}^{N} h_i(t_{i-1}, \nu_0)$$

Conclusion

In this paper, we have shown how to mathematically handle the pricing of a variance swap defined discretely with log returns, when the underlying asset follows a mean-reverting Gaussian model. Using log returns on a discrete grid is indeed one of the two typical ways of writing a variance swap on financial markets.

We saw that the valuation problem amounts to calculating N expectations, where N is number of points on the considered grid. Each one of the expectation can be turned into a simple analytical expression. To achieve such a result, we have considered a new pricing problem, which has then been split into two consecutive pricing problems on different intervals.

Finding a theoretical expression of the fair variance strike of a variance swap is important: the next step would consist in implementing the formula to see how it can be handled from a numerical point of view, and to compare its efficiency with other pre-existing methods when it comes to the valuation of variance swaps.

References

[1] P. Carr and D. Madan. Towards a theory of volatility trading, in volatility: New estimation techniques for pricing derivatives. *Risk publication*, 1998.

[2] M. Kamal K. Demeterfi, E. Derman and J. Zou. More than you ever wanted to know about volatility swaps. *Technical Report, Goldman Sachs Quantitative Strategies Research Notes*, 1999.

[3] P. Carr and A. Corso. Commodity covariance contracting. *Energy Risk*, 2001.

[4] S. L. Heston and S. Nandi. Derivatives on volatility: some simple solutions based on observables. *Working paper serier, Federal Reserve Bank of Atlanta*, 2000.

[5] T. Little and V. Pant. A finite-difference method for valuation of variance swaps. *Journal of Computational Finance*, 2001.

[6] P. Forsyth H. Windcliff and K. Vetzal. Pricing methods and hedging strategies for volatility derivatives. *Journal of Banking Finance*, 2006.

[7] S. P. Zhu and G. H. Lian. On the valuation of variance swaps with stochastic volatility. *Applied Math. Comput.*, 2012.

[8] S. P. Zhu and G. H. Lian. Pricing forward-start variance swaps with stochastic volatility. *Applied Math. Comput.*, 2015.

[9] X. C. Bi Z. L. Jia and S. G. Zhang. Pricing variance swaps under stochastic volatility with an ornstein-uhlenbeck process. *Journal of Systems Science and Complexity*, 2015.

[10] W. D. Zheng and Y. K. Kwok. Closed form pricing formulas for discretely sampled generalized variance swaps. *Mathematical Finance*, 2014.

[11] S. P. Zhu and G. H. Lian. A closed-form exact solution for pricing variance swaps with stochastic volatility. *Mathematical Finance*, 2011.

[12] E. M. Stein and J. E. Stein. Stock price distributions with stochastic volatlity and interest rate. *Review of Financial Study*, 1991.

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