Fractional Methods in Financial Modeling Part I : Introduction

Study carried out by the Quantitative Practice
Special thanks to Baptiste Chantry.

awalee

JUNE, 2023

Summary

Introduction

It is well known that the Black Scholes Merton model fails to reproduce some crucial features observed in the market of which the smile of volatility surface is the most obvious. One of these main issues lies in the modeling of the asset underlying which relies on one assumptions disconnected from the market obervations: the constant volatility.

The time series of the realized volatility go against these model assumptions and even reveal some complex features such as the trend to cluster in function of the spot underlying level. Stochastic models were introduced to deal with these inconsistencies, like the Heston model, a very popular model that was studied by both practioners and academics. In this model, both underlying asset and its variance are assumed to be stochastic processes driven by correlated Brownian motions.
Some of these models succeed to reproduce some complex features of the volatility surface particularly when a good calibration is performed. However, if some models can be a good choice to reproduce complex volatility surfaces, they suffer from over-fitting or fail to capture fine properties of the surface.
Indeed recent studies have proven that volatility is a persistence process and thanks to high frequency data more easily available and handled, it has been revealed that the regularity of its paths is slightly different from those generated by brownian motion. One limitation identified at he heart of these inconsistencies with models, seems to be the independance of the Brownian motion increments.

In consequence, we will focus our study on more general classes of processes as a source of randomness which can mainly reproduce the target features.
To capture all of them, we will consider "fractional" processes that is a class of continuous-time processes incorporating both roughness (irregular behavior at short time scales) and persistence (strong dependence at longer time scales). Hence we will introduce Volterra processes which is a general class of processes and particularly a subclass of processes called semistationary brownian processes.

As an example of these processes, we will focus intensively on fractional brownian motion (fBm), a generalization of the brownian motion which allows the correlation of increments. It will be very suitable to incorporate memory depending on a special value: the Hurst parameter.

The key features that make fractional processes very interesting regarding the modeling of volatility or interest rate processes includes the following: long-range memory, path dependence, non-Markovian propoerties, self-similarity, fractal paths. However, all these properties induces big challenges when it comes to design practical models with fractional processes.

Before diving into the power of Volterra processes, we introduce and recall some definitions and properties which will allow us to understand how the processes are defined properly. This will be the goal of this first note of the serie.

1 Fractional Processes

In this section, we will lay the foundations of the paradigm of fractional processes. Our main goal is to handle processes which can capture some common features like long range dependency.
After introducing the general framework and defining fundamental classes of processes, we will focus on a very particular and important case: the *fractional brownian motion.*

1.1 Volterra process

Let precise the framework in which we will developp our study:
for $T > 0$, let's consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions with a standard brownian motion $(W_t)_{t\in[0,T]}$ adapted to $\mathbb{F}$.

1.1.1 An introducing example

Consider the following Stochastic Differential Equation (SDE) with constants $\lambda, \eta > 0$:

$$dX_t = -\lambda X_t \mathrm{d}t + \eta \mathrm{d}W_t \qquad (1)$$

where W is a classical brownian motion and $X_0 \in \mathbb{R}$.

It could be shown that the stochastic process X defined by:

$$X_t = e^{-\lambda t} X_0 + \int_0^t \eta e^{-\lambda(t-u)} \mathrm{d}W_u \qquad (2)$$

is a (strong) solution of (1).

For a fixed s such as $0 \leqslant s \leqslant t$, we can write:

$$X_t = e^{-\lambda(t-s)} X_s + \int_s^t \eta e^{-\lambda(t-u)} \mathrm{d}W_u \qquad (3)$$

Taking a general mesurable function f and setting $\sigma = \eta\sqrt{\frac{1-e^{-2\lambda(t-s)}}{2\lambda}}$ we have:

$$\begin{aligned}
&\mathbb{E}\big[f(X_t)\,|\,\mathcal{F}_s\big] \\
&= \mathbb{E}\Big[f(e^{-\lambda(t-s)}X_s + \int_s^t \eta e^{-\lambda(t-u)}\mathrm{d}W_u)\,|\,\mathcal{F}_s\Big] \\
&= \mathbb{E}\Big[f(e^{-\lambda(t-s)}x + \int_s^t \eta e^{-\lambda(t-u)}\mathrm{d}W_u)\Big]\,|_{X_s=x} \\
&= \int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi}\sigma}exp(-\frac{(y-e^{-\lambda(t-s)}x)^2}{2\sigma^2})\mathrm{d}y|_{x=X_s}
\end{aligned}$$

we deduce that conditionally to $\mathcal{F}_s$:

$$X_t \sim \mathcal{N}\left(e^{-\lambda(t-s)}X_s, \eta^2\frac{1-e^{-2\lambda(t-s)}}{2\lambda}\right)$$

We can also derive the markovian property that is:

$$\mathbb{E}\big[f(X_t)\,|\,\mathcal{F}_s\big] = \mathbb{E}\big[f(X_t)\,|\,X_s\big]$$

This stochastic process is a Ornstein Uhlenbeck (OU) process which exposes a memory (mean reverting process) via the specific form of the exponential *kernel* e^{-t} depending on t.

1.1.2 Definition and first properties

Now let's be more general by replacing the exponential kernel by a kernel such as $K : [0,T]^2 \to [0,T]$ and consider the process:

$$X_t = g_0(t) + \int_0^t K(t,s)dW_s \tag{4}$$

with g_0 a bounded $\mathcal{F}_0$-measurable function. This process defines a *Volterra process.*

In the following, we assume that:

$$\sup_{t\in[0,T]}\int_0^t K(t,s)^2ds < \infty$$

Gaussian process A first result states that $(X_t)_{t\geqslant 0}$ is a gaussian process with mean X_0 and covariance function equals to

$$\mathbb{E}(X_tX_s) = \int_0^{\min(t,s)} K(t,u)K(s,u)\mathrm{d}u.$$

To derive a control on the moments of $(X_t)_{t\geqslant 0}$, the Burkholder-David-Gundy (BDG) inequality is the starting point and it states that:
for all $p > 0$, there exists universal constants $c_p, C_p < \infty$ such that, for all continuous local martingales $(X_t)_{t\geqslant 0}$ starting from zero, for all $T > 0$, the following inequalities hold in $[0;+\infty]$:

$$c_p\mathbb{E}(\sup_{t\in[0,T]}|X_t|^{2p}) \leqslant \mathbb{E}\big(\langle X\rangle_T^{p}\big) \leqslant C_p\mathbb{E}(\sup_{t\in[0,T]}|X_t|^{2p}) \tag{5}$$

Continuity From (5), we can show that assuming some boundary conditions, the Volterra process $(X_t)_{t\geqslant 0}$ has a continuous modification. Indeed, there exists a constant $N_p > 0$ such as:

$$\begin{aligned}&\sup_{t,s\in[0,T]}\mathbb{E}(|X_t - X_s|^p) \leqslant \\ &N_p\left(|g_0(t)-g_0(s)|^p + \Big(\int_s^t |K(t,s)|^2ds\Big)^{\frac{p}{2}} + \right.\\ &\left.\Big(\int_0^s (K(t,u)-K(s,u))^2\mathrm{d}u\Big)^{\frac{p}{2}}\right), \quad 0\leqslant s\leqslant t\leqslant T\end{aligned} \tag{6}$$

Then provided that, for a chosen γ:

$$\begin{aligned}&|g_0(t)-g_0(s)|^2 + \int_s^t K(t,s)^2\mathrm{d}s + \int_0^s (K(t,u)-K(t,u))^2\mathrm{d}u \\ &\leqslant c|t-s|^{\gamma}\end{aligned} \tag{7}$$

the result is a consequence of the application of the Kolmogorov criterion we recall below:
Let $(X_t)_{t\leqslant T}$ be a process in $\mathbb{R}$ with T >0 and such that: there exists $p\geqslant 1, \epsilon > 0$, and $c > 0$ such as:

$$\mathbb{E}\big(|X_t - X_s|^p\big) \leqslant c|t-s|^{1+\epsilon}, \quad t,s\leqslant T \tag{8}$$

Then there exists a modification of $(X_t)_{t\leqslant T}$ ie a process with continous paths whose are α-Hölder for all $\alpha \in \big[0, \frac{\epsilon}{p}\big)$. Hence, almost surely, for all $t\geqslant 0$, there exists a constant $\kappa\geqslant 0$ such as for all $u,v\in[0,t]$ then $|X_u - X_v| \leqslant \kappa|u-v|^{\alpha}$.

Moment estimates Following the use of the convexity of the function $x\mapsto |x|^p$ and the BDG inequality on the local martingale $u\mapsto \int_0^u K(t,s)\mathrm{d}W_s$, we can also derive moment estimates of the Volterra process $(X_t)_{t\geqslant 0}$ as follows:

$$\begin{aligned}&\sup_{t\in[0,T]}\mathbb{E}(|X_t|^p) \leqslant \\ &M_{p,T}\left(1 + \sup_{t\in[0,T]}\Big(\int_0^t K(t,s)^2\mathrm{d}s\Big)^{\frac{p}{2}}\right)\end{aligned} \tag{9}$$

1.2 Brownian semistationary process

Now let's introduce another class of processes: consider a stochastic process $X = (X_t)_{t\in\mathbb{R}}$ which admits the representation

$$X_t = \int_{-\infty}^t g(t-s)\sigma_s\mathrm{d}W_s, \qquad t\in\mathbb{R} \tag{10}$$

This is called a brownian semistationary process ($\mathcal{BSS}$). In (10) $\sigma = (\sigma_t)_{t\in\mathbb{R}}$ is a $(\mathcal{F}_t)$-adapted covariance-stationary process such as:

$$\sup_{t\in\mathbb{R}}\mathbb{E}[\sigma_t^2] < \infty \tag{11}$$

and g is a Borel-measurable function $g : (0,\infty) \to [0,\infty)$ with

$$\int_0^{\infty} g(x)^2\mathrm{d}x < \infty \tag{12}$$

The brownian motion is then convolved with this deterministic function g referred as the *kernel* in the following.
It is important to note that when σ is deterministic, X is gaussian while a stochastic σ makes it non-gaussian.
When W is independent of σ we have:

$$X_t|(\sigma_s)_{s\leqslant t} \sim \mathcal{N}\left(0, \int_0^{\infty} g(u)^2\sigma_{t-u}^2\mathrm{d}u\right)$$

Under some asumptions the process X is well defined and covariance-stationary. In *Fractional Methods in Financial Modeling part II*, we will introduce additional assumptions concerning the properties of the kernel g which will permit us to derive some crucial results for designing a method of simulation called hybrid-scheme. To be more precise we will focus specialy on processes of the form

$$X_t = \int_0^t g(t-s)\sigma_s \mathrm{d}W_s, \qquad t \in \mathbb{R} \tag{13}$$

also called as truncated Brownian semistationary process $\mathcal{TBSS}$. It is a particular case of Volterra process where $K(t,s) = g(t-s)\sigma_s$.

In the next parts of this serie, we will experiment the Monte Carlo pricing in models which incorporate this kind of process to fit the characteristics of the observed volatility smiles as we already discussed in 1. Indeed we will consider rough volatility models in which roughness will be driven by processes parts of $\mathcal{TBSS}$ class. In particular our equations of interest will have the following form in $\mathbb{R}$:

$$\begin{aligned} X_t = X_0 &+ \int_0^t g(t-s)b(X_s)\mathrm{d}s \\ &+ \int_0^t g(t-s)\sigma(X_s)\mathrm{d}W_s \end{aligned} \tag{14}$$

with $X_0 \in \mathbb{R}$, σ, b real continuous function with a linear growth condition that is:

$$|b(x)| + |\sigma(x)| < \kappa(1+|x|), \quad x \in \mathbb{R} \tag{15}$$

The part driven by the brownian motion is a special case of (13). The equation (14) is usually referred as a Stochastic Volterra Equation (SVE). As examples of kernel functions, we have:

- exponential kernel:

$$g(t) = e^{-\lambda t}, \quad \lambda > 0 \tag{16}$$

- gamma kernel:

$$g(t) = e^{-\lambda t}\frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \lambda > 0 \tag{17}$$

- fractional kernel:

$$g(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha \in \left(\frac{1}{2}, 1\right) \tag{18}$$

fractional regularity Now if we consider (7) again, in the particular case of the fractional kernel (18), we have $\gamma = 2H$ and we obtain that the paths of the fractional Volterra process are locally ϕ-Hölder continuous for $\phi \in (0, H)$. In this sense, H can be viewed as a roughness parameter. Note that in the case $H = \frac{1}{2}$, we recover the regularity of the brownian motion.

1.3 Fractional Brownian motion

In this section, we will focus on a very important example of a Gaussian Volterra process: the fractional brownian motion (fBm).

The Black Scholes model is maybe the most known pricing model that uses classical brownian motion as randomness source. Although it is a benchmark model, it is well known that it fails to reproduce some crucial features observed in the market. Among the reasons explaining this failing, the use of Brownian motion as randomness source needs to be highlighted.

Indeed, several studies show that the asset return does not have a Gaussian distribution law but exhibits an excess of Kustosis and heavy tails. Time series of return distribution also reveal a long-range dependency. From this observation, one solution would be to replace standard brownian motion with another process which exhibits such features. In this sense, the fractional Brownian motion is a perfect candidate specialy regarding the long range dependency.

Unfortunately, it has been shown that modelling asset return with diffusion driven by fBm can lead to arbitrage. Some corrections could erase these possibilities of arbitrage for example by adding in the diffusion a local martingale part driven by a brownian motion.

However recently the main focus changed and several studies suggest to use a persistent process like the fBm for modelling other processes which are involved in the pricing model like the spot variance or short rate processes. This is consistent with the option prices observed in the market and volatility time series. From this point, it is worth to study in details the properties of the fBm.

To clarify our study, let put the basis of the market model we will consider. Let $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$ be a filtered probability space with a filtration $\mathbb{F}$ generated by two independent Brownian motions W and $\tilde{W}$. The probability measure $\mathbb{Q}$ refers to one risk neutral measure.

Let $S = (S_t, t \in [0,T])$ be a strictly positive asset price process with the following dynamic:

$$dS_t = r_t S_t \mathrm{d}t + \sigma_t S_t(\rho \mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}\tilde{W}_t) \tag{19}$$

The volatility process σ is a square-integrable càdlàg (french acronym meaning "right continuous with left limit") process and we assume it exists a deterministic function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ such that:

$$\sigma_t = f(t, Y_t) \in L^1 \tag{20}$$

with Y a gaussian Volterra process such as:

$$Y_t = \int_0^t K(t,s)\mathrm{d}W_s$$

as previously presented.
Now we define the covariance function of Y_t: for all $t, s \geqslant 0$:

$$\Gamma(t,s) = \mathbb{E}(Y_s Y_t)$$

and the mean function:

$$m(t) = \mathbb{E}(Y_t)$$

Recall that a gaussian process is entierely defined with mean m and covariance function Γ, it can be shown that given a parameter called the Hurst parameter $H \in (0,1]$, it exists a centered gaussian process $W^H = (W_t^H)_{t\geqslant 0}$ with covariance function equals to

$$\Gamma(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t^{2H} - s^{2H}|) \tag{21}$$

for all $t, s \geqslant 0$. This process is called a *fractional brownian motion.*

From (21) we can derive the following properties which define it uniquely:

- $(W_{H,t})_0 = 0$ *as* and $\mathbb{E}(W_t^H) = 0$ for $t \geqslant 0$ (starting from zero)
- $(W_{H,t})_t$ has stationary increments
- $\mathbb{E}\left[(W_{H,t} - W_{H,s})^2\right] = |t-s|^{2H}$ for $t \geqslant s$

The sign of the covariance of the future and past increments is determined by the Hurst parameter. For $H > \frac{1}{2}$ the covariance is positive and for $H < \frac{1}{2}$ it is negative. Taking $H = \frac{1}{2}$ we get the classical brownian motion.

Another key result is the fact that for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ the fractional Brownian motion is not a semimartingale. Recall that given partition $\Pi = (t_i)_{i\geqslant n}$ such as $t_i \leqslant t_{i+1}$ with $\delta = \max_{i\leqslant n-1} |t_{i+1} - t_i|$ the *p-variation* of a stochastic process $(X_t)_{t\geqslant 0}$ is defined by:

$$v_p(\Pi) = \lim_{\delta \to 0} \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|^p$$

It can be demonstrated by showing that the total variation $(v_1(\Pi))$ and the quadratic variation $(v_2(\Pi))$ of the fBm is not finite except for $H = \frac{1}{2}$. And as we discussed earlier, it leads to the possibility of arbitrage opportunities when it is chosen as the randomness source for asset prices. Another difficulty from this property is the impossibility to use Îto Calculus.

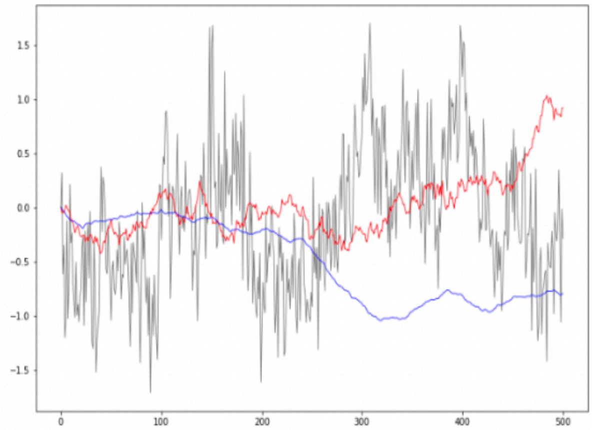

Figure 1: Simulated paths of fBm for different Hurst parameters: H=0.1 (grey), H=0.5 (red), H=0.8 (blue)

Integral representations What it will be particularly important for us is the integral representation of fBm. Among representations which exist, two specialy will be relevant for the following sections: the time representation and mainly the Volterra representations.

The time representation of fBm is such that:

$$W_t^H = \frac{1}{\kappa_H}\int_{-\infty}^{0}\left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right)dW_s + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s \tag{22}$$

with

$$\kappa_H = \sqrt{\frac{1}{2H} + \int_0^{+\infty}\left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds} < +\infty \tag{23}$$

The first Volterra representation is given by:

$$W_t^H = \int_0^t K(t,s)dW_s \tag{24}$$

with

$$K(t,s) = 1_{s<t}\frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H-\frac{1}{2})}{}_2F_1(H-\frac{1}{2}; \frac{1}{2}-H; H+\frac{1}{2}; 1-\frac{t}{s}) \tag{25}$$

with ${}_2F_1$ the hypergeometric function.

The second Volterra representation we will present here is the Rienmann-Liouville one. It will be our starting point for simulating the fBm (see *Fractional Methods in Financial Modeling part II*). This is given by:

$$W_t^H = \int_0^t K_H(t-s)dW_s \tag{26}$$

with

$$K_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}t^{H-\frac{1}{2}}, \quad H \in (0,1) \tag{27}$$

1.3.1 Long-Range dependency

In this section we will define and detail two important properties of fBm which are long-range dependency and self-similarity. Before explaining these properties let's consider the framwork of general stationary stochastic processes.

A stochastic process X is stationary if for all $n, d \geqslant 0$ and $k_1, ..., k_d \geqslant 0$, the vectors $(X_{k_1}, ..., X_{k_d})$ and $(X_{k_1+n}, ..., X_{k_d+n})$ have the same distribution. And since processes under our interest are mainly gaussian, it only requires that autocorrelations functions defined here by $\lambda(k) := \text{cov}(X_{n+k}, X_n)$ does not depend on n. Then the process X is said to be a process with long range dependency if its covariance functions $\lambda(.)$ are not integrable that is $\int_0^{\infty} |\lambda(s)| ds = \infty$. In a discrete view it is equivalent to require that covariance function decays slowly such that we have $\sum_{k=0}^{+\infty} \lambda(k) = \infty$ meaning intuitively the cumulative effect of the high-lag correlations is significant. Note that it exists other definitions of Long Range dependency based for example on slow varying function.

It is covenient to keep in mind that such a covariance structure have an impact on the statistical inference. For instance, taking n points of the process $X_1, ..., X_n$ when we assume $\text{Var}(X_i) < \infty$ for all $i \in [1, ..., n]$ then the variance of mean $\bar{X}_n = \frac{1}{n} \sum_{i=0}^{n} X_i$ is proportional to $\frac{1}{n}$ if the variables $X_1, ..., X_n$ are uncorrelated say independants. If we assume that $\lambda(k) = e^{-\alpha k}$, $\alpha > 0$ in this case, the autocorrelations functions are summable when n is large enough and the variance of the mean is still proportional to $\frac{1}{n}$.
On the other hand, if autocorrelations decay such as:

$$\lambda(k) \sim c|k|^{-\alpha}, \qquad k \to \infty \tag{28}$$

with $\alpha \in (0, 1)$, then autocorrelations are not summable. From a statistical view it modifies for instance the confidence interval of the mean $\bar{X}_n$ and all the tests related.

The covariance between increments at a distance $v = |t - s|$ decreases to zero as v^{2H-2} and in this way the fBm exhibits a long-range dependency.

It is a well established fact volatility is a persistent process and in this way, it makes sense to model this process with volatility models in which autocorrelation functions decay as (28). Traditionally models of (log) volatility which catch the long range dependency use fBm with an Hurst index $H = 1 - \alpha/2 \in (\frac{1}{2}, 1)$.

1.3.2 Self-Similarity

Quantifying longe range dependency can be an hard work and as an alternative, self-similar process can be used instead of quantifying this property. In this section we will define the self-similarity and its links with Longe Range dependency.

We say a stochastic process $(X_t)_{t \geqslant 0}$ is self-similar if it exists a real H such as: for all a,

$$X_{at} \stackrel{\mathcal{L}}{=} a^H X_t \tag{29}$$

with parameter H called the self-similarity exponent.

It is important to note a self-similar process cannot be a stationary process and in this case, Long Range Dependency is not under consideration. But this property rises if the increments are stationary and fBm is a perfect example of a self-similar process with stationary increments.

To have an overview of the relations between these properties and the nature of the process we can consider the figure below.

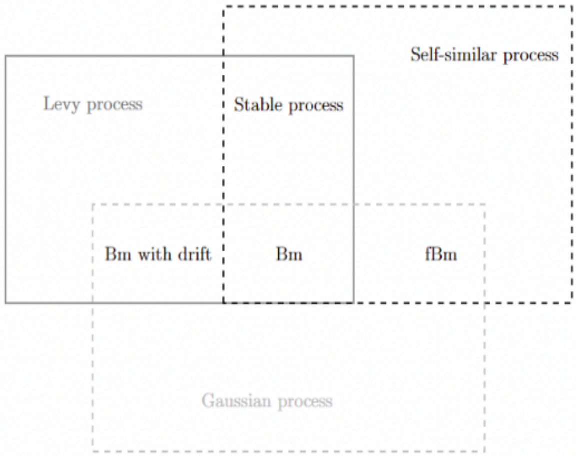

Figure 2: Relations between different types of processes and Long Range Dependency, self-similarity

Self similarity is defined by (29) and intuitively it show that a process have the same structure at different scales of time. But for a better understanding of the power law of self-similar process we can set $a = \frac{1}{t}$ to have:

$$X_t = t^H X_1$$

hence the CDF functions of X_t and X_1 noted respectively F_t, F_1 verify:

$$F_t(x) = F_1(\frac{x}{t^H})$$

leading:

$$f_t(x) = \frac{1}{t^H} f_1(\frac{x}{t^H})$$

where f_t and f_1 are the densities respectively of X_t and X_1. By setting $x = 0$, we get:

$$f_t(0) = \frac{1}{t^H} f_1(0) \tag{30}$$

showing up that the density of X_t is the same as X_1 modulo a scaling term.

In order to check the self-similarity property, there are two approaches: the first is by using (30) for an estimation of the Hurst parameter. This approach needs to estimates H by regression after estimating $f_t(0)$ using kernel estimator or empirical histogram. The second is called the curve fitting method and is based on comparing the aggregation properties of empirical densities.

2 Fractional Derivative

The starting point of this section is the definition of a specific operator. It could seem to be not related with the subject at first glance but such operator will arise in next section when we will focus on pricing model including Volterra process. Since this note aims to group all the basics of the serie, it has been chosen to include this definition at this stage.

In order to illustrate the araising of such operator and to link it with pricing issues, we will give the example of the rough version of the famous Heston model. In this case, characteristic function of the log price $\mathbb{E}[e^{u\log(S_t)}]$, $u \in i\mathbb{R}$, of the underlying (stock) S is expressed in terms of functions depending on the solution to a specific "fractional Riccati equation" in form of $D^\alpha \Phi = \mathcal{P}(\Phi)$ where $\mathcal{P}$ is a second degree polynom. In this section we will attach to give a precise sense to the operator D^α.

Recalling that $F : I \to \mathbb{R}$ with $I \subset \mathbb{R}$ is called a primitive function of $f : I \to \mathbb{R}$ if for $x \in I$, $\frac{dF(x)}{dx} = f(x)$, the fundamental theorem states if $a \in \mathbb{R}$ the function defined as $x \to \int_a^x f(x)dx$ is a primitive function of f and if D is the derivative operator we have:

$$D(F)(x) = f(x) \tag{31}$$

and we can see the integration as the reverse operation of differentiation.

Then one can define the n-th derivative operator by $D^n(f) = \frac{d}{dx}(D^{n-1}(f))$. Applying the same process to integration permit us to define what we call the integral of n-th order represented by D^{-n}:

$$D^{-n}(f)(x) = \int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-1}} f(y_n)dy_n \dots dy_2 dy_1 \tag{32}$$

We thus have the following direct result:

$$D^m(D^{-n}(f)) = D^{m-n}(f), m > n$$

Let f be an integrable function on $I = [a, b]$ with $a, b \in \mathbb{R}$. Let $x \in I$. Then, we have: for all $n \geq 2$,

$$\int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-2}} \int_a^{y_{n-1}} f(y_n)dy_n dy_{n-1} \dots dy_1 = \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u)du$$

This result is our starting point. Indeed, the natural following step is to generalize this result to real number, and using the Gamma function it leads to what we call the Riemann-Liouville fractional integral:

$$\frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} f(u)du, \quad \alpha > 0$$

It will be discussed the conditions under which this integral is well-defined.

Let now define the α-order derivative. The process is the following: we can define $D^{-\alpha}$ for any $\alpha > 0$ and we can define D^n for any integer $n \geqslant 1$. Hence, taking $n - 1 < \alpha < n$, we can define the fractional integral $D^{n-\alpha}$. The next step is to take the n-th order derivative of $D^{n-\alpha}$ then, it defines the (left-hand) fractional derivative:

$$D^\alpha f = \frac{d^\alpha f}{dx^\alpha} = \frac{d^n}{dx^n}(D^{n-\alpha} f)$$

2.1 Rienmann Liouville Fractional Derivative

We have seen the strategy used to define fractional integral and derivative operator. We now define the Rienmann-Liouville (RL) fractional derivative whcih is more general and will be very important insofar as it is one way to represent the fractional Brownian motion. The next section will be the object of study of this stochastic process.

Let $I = [a, b]$ be a finite real interval. We define the left-side fractional integral by:

$$I_{a^+}^\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad a < x, 0 < \alpha \tag{33}$$

As a remark, we can define in the same way the right side Rienmann-Liouville fractional integral:

$$I_{b-}^\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x < b, \ 0 < \alpha$$

With $n \in \mathbb{N}$, setting $\alpha = n$, we have:

$$I_{a^+}^\alpha(f)(x) = \frac{1}{(n-1)!} \int_a^x f(t)(x-t)^{n-1} dt, \quad a < x, \ 0 < \alpha \tag{34}$$

We can now define the left side fractional derivative of order $\alpha > 0$. Setting $n = \lfloor \alpha \rfloor + 1$:

$$D_{a^+}^\alpha(f)(x) = D^n(I_{a^+}^{n-\alpha}(f)) = \left(\frac{d^n}{dx^n}\right) \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1+\alpha-n}} dt, \quad a < x, \ 0 < \alpha \tag{35}$$

Let $n \in \mathbb{N}$ then we can verify that: $D_{a^+}^0(f)(x) = f(x)$ and $D_{a^+}^n(f)(x) = f^{(n)}(x)$. Considering a constant function $f \equiv c \in \mathbb{R}$ on a finite interval $I = [a, b] \subset \mathbb{R}$. Let $\alpha = \frac{p}{q} \in (0, 1)$ be a fractional number. In the figure 1 below we see the graph of the fractional derivative function *cos* for different orders.

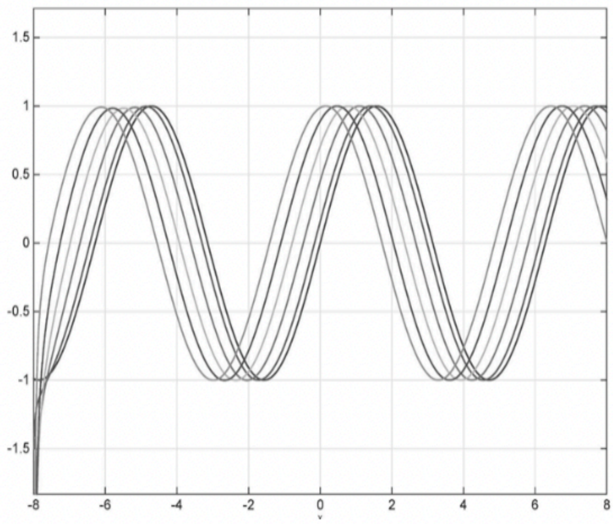

Figure 3: Rienmann-Liouville fractional derivative of *sin* fuction for order $\alpha = 0, 0.1, \ldots, 0.9$ corresponding of the graphs from right to left

2.2 Caputo and others Fractional Derivative

Several other definitions of fractional derivative exist. Among them we can cite: Marchaud derivative and left/right Grünwald-Letnikov derivative. Some others are also based on the Riemann-Liouville integral like the Caputo which is similar in the form to the RL one. Setting $n = \lfloor \alpha \rfloor + 1$ i is defined as follows:

$$\begin{aligned} D^{\alpha}_{a^+}(f)(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{1+\alpha-n}} dt \\ &= I_a^{n-\alpha}\left(\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right)(x), \ a < x, \ 0 < \alpha, \end{aligned} \tag{36}$$

In the next notes of the serie, we will consider the RL representation of the fractional derivative.

Conclusion

The volatility surface exhibits some features like persistence and roughness. In this first note of the serie dedicated to fractional methods, we have seen that in order to reproduce these properties, special classes of stochastic process are available. In this sense, we presented the volterra processes which is a general class defined on a what we called a kernel function. Particularly we have highlighted a subclass of Volterra process named truncated Brownian semistationary processes.

A special case belonging to this class, is the fractional brownian motion: it admits a Volterra representation based on a special kernel in the form of a convolution with a brownian motion. This kernel is called the fractional kernel and it permits to bring the roughness we are interested in. We have also seen that its properties like long range dependency and self similarity are crucial and made it a tremendous source of randomness for modelling process like volatility.

In the second part, we introduced the important and non-intuitive notion of fractional derivative operator which will arise in the rough volatility models we will consider in the following notes. Since this operator is rarely defined in details, the main purpose was to clarify and precise this key operator.

References

[1] G. ANdersen and al. Modeling and forecasting realized volatility. *Econometrica*, 2003.

[2] Varsha Daftardar-Gejji. *Fractional Calculus and Fractional Differential Equation.* Birkhäuser, 2019.

[3] Thierry Meyre Francis Comets. *Calcul stochastique et modèles de diffusions.* Editions DUNOD, 2015.

[4] Eduardo Abi Jaber and al. Affine volterra processes. *np*, 2019.

[5] L. C. G. Rogers. Arbitrage with fractional brownian motion. *np*, 1997.

[6] Cheridito. Gaussian moving averages, semimartingales and option pricing. *Stochastic processes and their applications*, 2004.

[7] T. Hayashi and N. Yoshida. On covariance estimation of non-synchronously observed diffusion processes. *Bernoulli*, 2005.

[8] Brollerslev and al. Arch modeling in finance. *ournal of Econo- metrics*, 2003.

A propos d'Awalee

Cabinet de conseil indépendant spécialiste du secteur de la Finance.

Nous sommes nés en 2009 en pleine crise financière. Cette période complexe nous a conduits à une conclusion simple : face aux exigences accrues et à la nécessité de faire preuve de souplesse, nous nous devions d'aider nos clients à se concentrer sur l'essentiel, à savoir leur performance.

Pour accomplir cette mission, nous nous appuyons sur trois ingrédients : habileté technique, savoir-faire fonctionnel et innovation.

Ceci au service d'une ambition : dompter la complexité pour simplifier la vie de nos clients.

«*Run the bank*» avec Awalee !

Contactez-nous

Ronald LOMAS
Partner
rlomas@awaleeconsulting.com
06 62 49 05 97