Funding Valuation Adjustment with the Reloaded Least Squares Monte Carlo

awalee

Summary

Introduction

The introduction of valuation adjustments is a great financial dilemma both for the understanding of these adjustments and their modeling, but also for the complexity of their calculation. The Funding Valuation Adjusmtment (FVA) is without a doubt, one of the valuation adjustments that best characterizes this dilemma. It is rapidly become the standard to evaluate the funding cost.

The traditionnal method used by the banks to calculate the XVAs is the so-called Least Square Monte Carlo (LSM). This method is introduced by Longstaff and Schwartz [1] for the american option calculation. Cesari et. al. [2] has adapted the LSM to the calculation of the XVA.

We consider the Funding Valuation Adjustment (FVA) equation arising from the funding invariance principal. We show that the FVA equation can be written as the difference of two prices discounted with different rates. We show that in a gaussian framework, when the transaction does not implied any exchange of collateral the FVA can be priced easily as differents of two prices calculated by analytical formula. However when the transaction involves an exchange of collateral, we introduced the reloaded Least Square Monte (LSM) Carlo to approximate the non-linearity of the collateral. Then the FVA equation is approximated as difference of two prices where one of the price involves the estimation of the collateral non-linearity by the reloaded LSM method.

The paper is organized as follows. In Section 1 we state our working assumption, notations used throughthe paper and we introduced the FVA formula. Section 1.1 presents the reloading LSM. In Section 1.2 we applied the relaoading LSM to the FVA calculation. Finally, Section 2 shows some numerical test.

1 FVA modelling

Let assume a pricing stochastic basis $(\Omega; \mathbb{G}; \mathbb{P})$ with a model filtration $\mathbb{G} = (G_t)_{t\in\mathbb{R}_+}$ and risk-neutral pricing measure $\mathbb{P}$ such that all the processes of interest are $\mathbb{G}$ adapted. We consider a transaction in an OTC market between a bank and a counterparty. We denote by X the product cash flow of the transaction and V the risk-neutral conditional expectation of its future discounted promised cashflows (mark-to-market) and Φ is the collateral posted by the bank. The collateral is modelled as a non-linear function of the mark-to-market V. We write r^C for the remunerated rate of the collateral posted, r^F for the funding rate. Their corresponding discount factor processes at a time $t > 0$ are denoted by β_t^C, β_t^F, where $\beta_t^* = e^{\int_0^t -r_s^* ds}$. We denote by $s^F = r^* - r^F$ and $s^C = r^* - r^C$ respectively the funding and the remunerated spread. We assume that all interest rate processes are stochastics model by a Gaussian framework and that all spreads are deterministic functions.

Starting with the default-free framework we consider all cashflows including collateral and funding swap whitout the default risk cashflows. Thereby, the master pricing equation (default free case) of the transaction is given as follows:

$$V_t = \mathbb{E}\Big[\int_0^T \beta_t^* dX_t + \int_0^T \beta_t^* s^C \Phi(V_t)dt + \int_0^T \beta_t^* s^F (V_t - \Phi(V_t))dt\Big] \tag{1.1}$$

with $V_t^* = \mathbb{E}_t\left[\int_t^T \beta_s^* \frac{dX_s}{\beta_t^*}\right]$.

The first term of the above equation represent the value of the derivative transaction without default risk and funding account, the second term represent the CSA funding and the last term is treasure funding of the bank.

The funding invariance principle (see Elouerkhaoui [3]) states that we can discounted the cash-flows of the trade (inclusive of CSA and Treasure funding) with any rate that we choose. Then, we have the following equivalence of the equation (1.1):

- CSA discounting ($r^* = r^C$)

$$V_t = \mathbb{E}\Big[\int_0^T \beta_t^C dX_t + \int_0^T \beta_t^C (r^C - r^F)(V_t - \Phi(V_t))dt\Big]$$

- Funding discounting ($r^* = r^F$)

$$V_t = \mathbb{E}\Big[\int_0^T \beta_t^F dX_t + \int_0^T \beta_t^F (r^F - r^C)\Phi(V_t)dt\Big]$$

The FVA corresponds to the funding part (CSA and treasure) of the equation (1.1) and is given by the following equation:

$$\text{FVA}_0 = \mathbb{E}\left[\int_0^T \beta_t s^C \Phi(V_t^F)dt + \int_0^T \beta_t s^F (V_t^F - \Phi(V_t^F))dt\right], \tag{1.2}$$

with $V_t^F = \mathbb{E}_t\left[\int_t^T \beta_s^F \frac{dX_s}{\beta_t^F}\right]$.

Starting from this formulation we aim to represent the FVA as the difference of a funding price at the rate r^F, V^F, and the funding price at the rate r, $V_t = \mathbb{E}_t\left[\int_t^T \beta_s \frac{dX_s}{\beta_t}\right]$.

In Section 1.1 we recall the reloading LSM. In Section 1.2 we applied the reloading LSM to the FVA calculation.

1.1 Recall the reloading LSM

The reloaded LSM is adapted to the problems of calculating XVAs by Huge and Savine [4]. This approach is called *Proxies Only in Indicators* (POI).

For a straightforward presentation of this method we focus on the CVA calculation with no collateral and no marge with a discount factor set to 1. We consider the following equation:

$$\text{CVA}_0 = \mathbb{E}\left[1_{0<\tau_C<T}(V_{\tau_C})^+\right], \tag{1.3}$$

where τ_C is the default time of the client, and $V_t = \mathbb{E}_t\left(\int_t^T dX_s\right)$ with X_s the flow generated by the deal.

The main idea of this approach is instead of doing classical LSM on V, to use an approximation of the nonlinearity of the

CVA represented by $1_{V>0}$. To this end the equation (1.3) is rewrite as follows:

$$\begin{aligned}\mathrm{CVA}_0 &= \mathbb{E}_0\left[1_{0<\tau_C<T} V_{\tau_C} 1_{V_{\tau_C}>0}\right] \\ &= \mathbb{E}_0\left[1_{0<\tau_C<T}\mathbb{E}_{\tau_C}\left(\int_{\tau_C}^T dX_s\right) 1_{V_{\tau_C}>0}\right] \\ &= \mathbb{E}_0\left[\int_0^T 1_{0<\tau_C<T} 1_{V_{\tau_C}>0} dX_s\right].\end{aligned} \tag{1.4}$$

Then we approximate $1_{V_{\tau_C}>0}$ by $1_{\tilde{V}_{\tau_C}>0}$ where $\tilde{V}$ is an approximation of V obtain by regression. We obtain the following approximation of the CVA:

$$\mathrm{CVA}_0 \approx \mathbb{E}\left[\int_0^T 1_{0<\tau_C<T} 1_{\tilde{V}_{\tau_C}>0} dX_s\right]. \tag{1.5}$$

It is intended to be more precise than the classic LSM. In fact in the classical LSM that involves the following approximation

$$\mathrm{CVA}_0 \approx \mathbb{E}\left[1_{0<\tau_C<T} \tilde{V}^+_{\tau_C}\right],$$

we used an biased proxy of price, that involves an precision issue, however the POI approach uses the right payoff that lead to a better precision. Figure 1 shows the difference of the error between LSM and POI in a case of a call option. We show that with the POI the pricing is done with the right strike, the only mistake being to exercise the option out of money in certain scenarios.

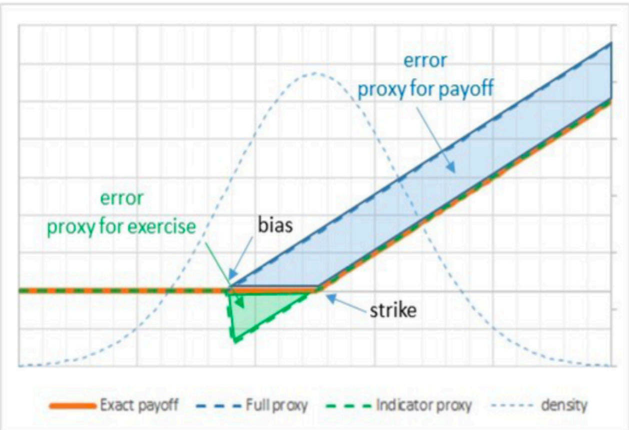

Figure 1: Figure taken from [4]

1.2 FVA calculation using reloading LSM

Section 1.2.1 shows an adaptation of the reloading LSM in a case without collateral. Section 1.2.2 shows, the main contribution of this note, the adaptation of the reloading LSM in a case with collateral.

1.2.1 Case without collateral

First we consider and OTC transaction without CSA, the collateral Φ is vanished in the equation (1.2). In this case the equation (1.2) is reduced to:

$$\mathrm{FVA}_0 = \mathbb{E}\left[\int_0^T \frac{r_t^* - r_t^F}{\beta_t^*} V_t dt\right], \tag{1.6}$$

where the mark-of-market V_t is defined as:

$$V_t^F = \mathbb{E}_t\left[\int_t^T \beta_t^F \frac{dX_u}{\beta_u^F}\right]. \tag{1.7}$$

Proposition 1. *If $\Phi = 0$, then (under some technical hypothesis highlighted in the proof)*

$$\mathrm{FVA}_0 = V_0^F - V_0. \tag{1.8}$$

Proof. First we replace V_t^F by its formula in the equation (1.6),

$$\mathrm{FVA}_0 = \mathbb{E}\left[\int_0^T \underbrace{\frac{s_t^F}{\beta_t^*}\mathbb{E}_t\left[\int_t^T \beta_t^F \frac{dX_u}{\beta_u^F}\right]}_{f(u,t)} dt\right], \tag{1.9}$$

and assume that $\mathbb{E}\left[\int_0^T \mid f(u,t) \mid dt\right] < \infty$, we can apply the Fubini-Tonelli theorem by switching the expectation and the lebesgue integral. We obtain,

$$\mathrm{FVA}_0 = \int_0^T \mathbb{E}\left[\underbrace{s_t^F \frac{\beta_t^F}{\beta_t^*}}_{U} \mathbb{E}_t\left[\underbrace{\int_0^T 1_{u>t} \frac{dX_u}{\beta_u^F}}_{V}\right]\right] dt \tag{1.10}$$

and by considering that U is G_t measurable and limited then $\mathbb{E}\left[U\mathbb{E}_t\left[V\right]\right] = \mathbb{E}\left[UV\right]$ which gives,

$$\begin{aligned}\mathrm{FVA}_0 &= \mathbb{E}\left[\int_0^T s_t^F \frac{\beta_t^F}{\beta_t^*} \int_0^T 1_{u>t} \frac{dX_u}{\beta_u^F} dt\right] \\ &= \mathbb{E}\left[\int_0^T \int_0^T \underbrace{s_t^F \frac{\beta_t^F}{\beta_t^*} 1_{u>t} \frac{1}{\beta_u^F}}_{\gamma((u,t)} dX_u dt\right]\end{aligned} \tag{1.11}$$

Assume that $\int_0^T \left(\int_0^T \mid \gamma(u,t) \mid^2 du\right)^{\frac{1}{2}} dt < \infty$ almost surely, we can apply the stochastic Fubini theorem (see Neerven and Veraar [5]) to interchange the Lesbesgue and the stochastic integral. We obtain,

$$\mathrm{FVA}_0 = \mathbb{E}\left[\int_0^T \int_0^T s_t^F \frac{\beta_t^F}{\beta_t^*} 1_{u>t} dt \frac{dX_u}{\beta_u^F}\right] \tag{1.12}$$

Finally, by some simple calculs we obtain,

$$\begin{aligned}\mathrm{FVA}_0 &= \mathbb{E}\left[\int_0^T \int_0^u s_t^F e^{-\int_0^t s_r^F dr} dt \frac{dX_u}{\beta_u^F}\right] \\ &= \mathbb{E}\left[\int_0^T \left(1 - e^{-\int_0^u s_r^F dr}\right) \frac{dX_u}{\beta_u^F}\right] \\ &= \mathbb{E}\left[\int_0^T \frac{dX_u}{\beta_u^F}\right] - \mathbb{E}\left[\int_0^T e^{-\int_0^u s_r^F dr} \frac{dX_u}{\beta_u^F}\right] \\ &= \mathbb{E}\left[\int_0^T \frac{dX_u}{\beta_u^F}\right] - \mathbb{E}\left[\int_0^T \frac{dX_u}{\beta_u^*}\right]\end{aligned} \tag{1.13}$$

□

The equation (1.8) allows to overcome the numerical challenge of FVA computation. In fact the complexity returns to a pricing of derivative product.

1.2.2 Case with collateral

In this section we generalized the proposition 1 in the case where Φ is not vanished. In pratice the collateral can be modelled as a call spread function ($\Phi(V_t) = (V_t - Th^{up})^+ - (V_t - Th)^-$). For the sake of simplicity, we assume that $\Phi(V) = (V)^+$. The equation (1.2) can be rewrited as follows:

$$\mathrm{FVA}_0 = \underbrace{\mathbb{E}\left[\int_0^T \frac{r_t^* - r_t^F}{\beta_t^*} V_t dt\right]}_{I} - \underbrace{\mathbb{E}\left[\int_0^T \frac{r_t^C - r_t^F}{\beta_t^*} \Phi(V_t) dt\right]}_{II}, \tag{1.14}$$

where the mark-of-market V_t is defined as:

$$V_t^F = \mathbb{E}_t\left[\int_t^T \beta_t^F \frac{dX_u}{\beta_u^F}\right]. \tag{1.15}$$

Proposition 2. *In the case with collateral $\Phi(V_t^F)$, we have (under technical assumptions similar to those specified in the proof of the Proposition 1)*

$$\mathrm{FVA}_0 = \mathbb{E}\left[\int_0^T \hat{\beta}_u dX_u\right] - V_0 \tag{1.16}$$

où $\hat{\beta}_u = \beta_u^F \left(1 - \int_0^u (r_t^C - r_t^F) e^{-\int_0^t (r_w^C - r_w^F) dw} \frac{1_{V_t > 0}}{e^{\int_0^t s_w^C dw}} dt\right)$.

Proof. By the proposition 1 we have

$$I = V_0^F - V_0^*.$$

By using the same mathematical tools as in the proof of the proposition 1, we have,

$$II = \mathbb{E}\left[\int_0^T \frac{r_t^C - r_t^F}{\beta_t^*} 1_{V_t > 0} \int_0^T 1_{u > t} \beta_t^F \frac{dXu}{\beta_u^F} dt\right] \tag{1.17}$$
$$= \mathbb{E}\left[\int_0^T \int_0^u (r_t^C - r_t^F) e^{-\int_0^t (r_w^C - r_w^F) dw} \frac{1_{V_t > 0}}{e^{\int_0^t s_w^C dw}} dt \frac{dXu}{\beta_u^F}\right].$$

Finally by adding I and II we have:

$$\mathrm{FVA}_0 = \mathbb{E}\left[\int_0^T \left(1 - \int_0^u (r_t^C - r_t^F) e^{-\int_0^t (r_w^C - r_w^F) dw} \frac{1_{V_t > 0}}{e^{\int_0^t s_w^C dw}} dt\right) \frac{dXu}{\beta_u^F}\right] - V_0^* \tag{1.18}$$

□

2 Numerical test

We present in this section some numerical examples. Let consider a portfolio of long position of call spread on FX^{ij}, where i is the domestic currency and j is the foreign currency. We aim to calculate the value of the FVA in this transaction.
We consider a HJM multifactor model to modelize the rate and FX. For two currencies i, j, the exchange rate FX^{ij} is modelised, for $t > 0$, by:

$$\frac{dFX^{ji}(t)}{FX^{ji}(t)} = (r_t^i - r_t^j)dt + \langle \Gamma_{ji}^{FX}, dW_t^i \rangle, \tag{2.1}$$

where r_t^i, r_t^j are the spot rate of the economies i and j, Γ_{ji}^{FX} is the deterministic volatility vector and W^i is a Brownian motion under the measure $\mathbb{P}^i$ associated to the numeraire $(\beta^i)^{-1}$.

The spot rate r^i is associated to the forward rate f^i by the relation $r_t^i = f^i(t,t)$, where

$$df^i(t,T) = \langle \sigma_i(t,T), \Gamma_i(t,T) \rangle dt + \langle \sigma_i(t,T), dW_t^i \rangle, \tag{2.2}$$

with $\sigma_i(t,T)$ and $\Gamma_i(t,T)$ deterministics vectors and W^i and Brownian motion under $\mathbb{P}^i$. To express the dr_t^j dynamic under the domestic currency a convexity adjustment is then apply:

$$dW_t^j = dW_t^i - \Gamma^{ji}(t)dt.$$

We also introduced the discount rate curve (or *discount* d) : the neutral risk rate r, the funding rate r^F and the collateral remuneration rate posted by r^C. We modelized these rates by the following Vasicek model:

$$dr_t^d = \alpha(\theta - r_t^d)dt + \langle \Gamma_t^d, dW_t^d \rangle, \tag{2.3}$$

whith α the speed of reversion and θ the long term mean level. We consider that the spread of funding and the spread of collateral remuneration $s^F = r - r^F$ and $s^C = r - r^C$ are constants. The value of a portfolio of call spread FX discounting at the rate r^d is given at time t by:

$$V_t^d = \sum_{j=1}^N (-\mathbb{E}_t\left[e^{-\int_t^T r_u^d du}\left(FX_T^{ij} - K_1\right)^+\right] \tag{2.4}$$
$$+\mathbb{E}_t\left[e^{-\int_t^T r_u^d du}\left(K_2 - FX_T^{ij}\right)^+\right]),$$

with N the number of currency, i the domestic currency, K_1 et K_2 the strike, T the maturity.

As we are in a gaussian word we can calculated V_t^d by analytical formula.

2.1 Case without collateral

We recall the formula (1.6)-(1.8) of the FVA of FX^{ij} call spread transaction when the collateral is vanished:

$$\mathrm{FVA}_0 = \mathbb{E}\left[\int_0^T \beta_t(r_t - r_t^F)V_t^F dt\right] = V_0^F - V_0,$$

where V correspond to V^d for $r^d = r$ in (2.4).

The figure 2 shows the evolution of the relative error of the FVA (without collateral) calculated by (1.6) and (1.8) with various maturities. This error is given with respect to the number of MC trajectories. We show that the maturity of the deal play an important role on the convergence of the MC method. For a short maturity($T = 0.2$) the relative error is of order of 0.01 by generating 10^3 MC trajectories while for a long maturity ($T = 1$) the relative error is 0.2 for the same number of trajectories.

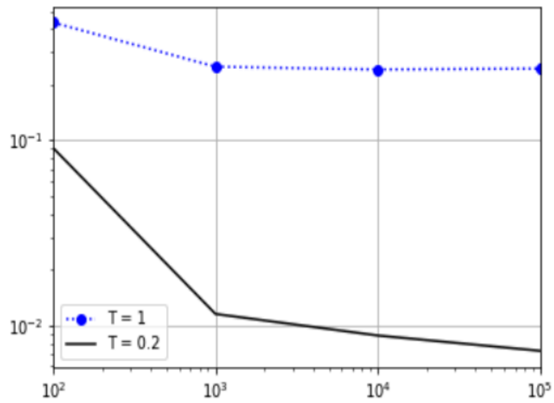

Figure 2: Relative error of FVA (without collateral) with respect to the number of MC trajectories: $N = 2$, $FX_0 = 1$, $r_0 = 0.1$, $s^F = 0.001$, $K_1 = 0.95$, $K_2 = 1.2$.

The figure 3 shows the evolution of the relative error of the FVA (without collateral) calculated by (1.6) and (1.8) with diferents number of currencies. This error is given with repect to the number of MC trajectories. We show that the number of currency have an impact to the convergence of the approximation.

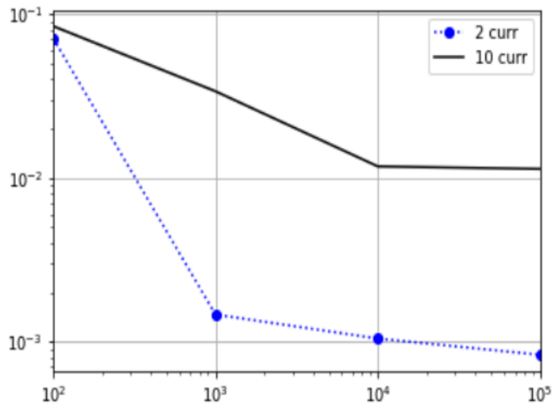

Figure 3: Relative error of FVA (without collateral) with respect to the number of MC trajectories. $FX_0 = 1$, $r_0 = 0.1$, $s^F = 0.001$, $K_1 = 0.95$, $K_2 = 1.2$, T = 0.2.

2.2 Case with collateral

We consider in the following a deal of a FX call spread. In the collateral framework, the FVA is formalized as follows:

$$\mathrm{FVA}_0 = \mathbb{E}\left[\int_0^T \beta_t (r_t - r_t^F) V_t^F - \beta_t (r_t^C - r_t^F)\Phi(V_t^F) dt\right], \quad (2.5)$$

$$= \mathbb{E}\left[\hat{\beta}\sum_{j=1}^{N}\left(-\left(FX_T^{ij} - K_1\right)^+ + \left(K_2 - FX_T^{ij}\right)^+\right)\right] - V_0,$$

where

$$\hat{\beta} = \beta_T^F\left(1 - \int_0^T (r_t^C - r_t^F) e^{-\int_0^t (r_w^C - r_w^F)dw}\frac{1_{V_t>0}}{e^{\int_0^t s_w^C dw}} dt\right).$$

We used the POI method,the indicator $1_{V_t>0}$ that appears in $\hat{\beta}$ is approximated by $1_{\tilde{V}_t>0}$ where $\tilde{V}$ is an approximation of V by regression. Then the FVA is approximated by:

$$\mathrm{FVA}_0 \approx \mathbb{E}\left[\hat{\beta}\sum_{j=1}^{N}\left(-\left(FX_T^{ij} - K_1\right)^+ + \left(K_2 - FX_T^{ij}\right)^+\right)\right] - V_0, \quad (2.6)$$

where $\hat{\beta} \approx \beta_T^F\left(1 - \int_0^T (r_t^C - r_t^F) e^{-\int_0^t (r_w^C - r_w^F)dw}\frac{1_{\tilde{V}_t>0}}{e^{\int_0^t s_w^C dw}} dt\right)$.

The Figure 4 show the relative error of the FVA with collateral calculated by the POI method (Left) and by using the LSM method (Right). This error is given with respect to the number of MC trajectories. We used all the risk factor and their square as regression basis. We show that the POI method has a relative error smaller than the one with LSM. However, this method does not work well with hight dimension and long maturities.

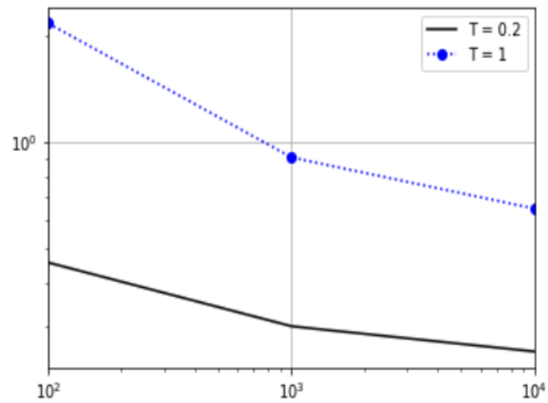

Figure 4: Relative error of FVA (without collateral) with respect to the number of MC trajectories: $FX_0 = 1$, $r_0 = 0.1$, $K_1 = 0.95$, $K_2 = 1.2$, $s_F = 10bps$, $s_C = 5bps$: Regression by POI.

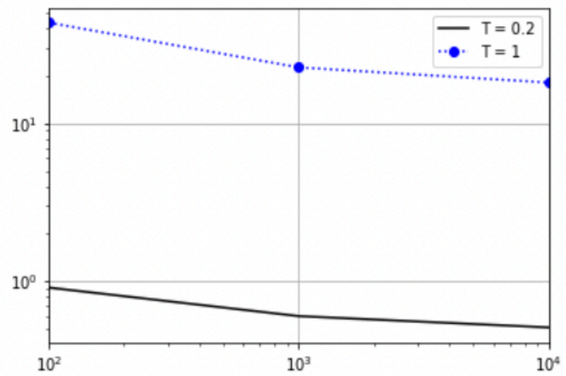

Figure 5: Relative error of FVA (without collateral) with respect to the number of MC trajectories: $FX_0 = 1, r_0 = 0.1, K_1 = 0.95, K_2 = 1.2, s_F = 10bps, s_C = 5bps$: Régression classic.

Conclusion

We consider the Funding Valuation Adjustment (FVA) equation arising from the funding invariance principal. Whe show that the FVA equation can be written as the difference of two prices discounted with different rates. We show that in a gaussion framework, when the transaction does not implied any exchange of collateral the FVA can be priced easily as differents of two prices calculated by analytical formula. However when the transaction involves an exchange of collateral, we introduced the reloaded Least Square Monte (LSM) Carlo to approximate the non-linearity of the collateral. Then the FVA equation is approximated as difference of two prices where one of the price involves the estimation of the collateral non-linearity by the reloaded LSM method. We show that using the reloaded Least Square Monte Carlo also called Proxy on Indicator is more accurate than the so-called Least Square Monte Carlo.
Numerical methods based on regression (LSM, POI) are well suited for the calculation of first generation XVAs (CVA, FVA). However, second generation XVAs (MVA, KVA) involving the computation of conditional risk measures in the future need to explore other methods such as the Nested Monte Carlo.

A propos d’Awalee

Cabinet de conseil indépendant spécialiste du secteur de la Finance.

Nous sommes nés en 2009 en pleine crise financière. Cette période complexe nous a conduits à une conclusion simple : face aux exigences accrues et à la nécessité de faire preuve de souplesse, nous nous devions d’aider nos clients à se concentrer sur l’essentiel, à savoir leur performance.

Pour accomplir cette mission, nous nous appuyons sur trois ingrédients : habileté technique, savoir-faire fonctionnel et innovation.

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