Gatheral's Double Log Normal Stochastic Volatility Model For Calibration

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Introduction

The VIX is an average of out-of-money option prices accros all available strikes on S&P 500 index. The goal of the VIX index is to capture the volatility of the S&P500 over the next-month implicit in stock index option prices. In practice, the VIX can be replicated by a T-expiry log contract [1, 2, 3]. However, perfect replication requires frictionless markets and continuity of the price process. Thus, the authors in [4] have introduced the theoretical proxy of VIX and demonstrated the necessity of employing a model.
Nowaday, it is possible to directly invest in volatility as an asset class by means of VIX derivatives. Specifically, trading in futures on the VIX began on a decade ago. There are standard futures contracts on forward 30-day implied volatility that cash settle to a special opening quotation of the VIX on the Wednesday that 30 days prior to the 3^{rd} Friday of the calendar month immediately following the expiring month.
Numerous research has been drawn to the development of appropriate pricing model for VIX futures. Carr & Wu [5] presented a lower and upper bounds for the price of VIX futures. In fact, under some assumptions, the authors showed that the lower bound for the VIX futures can be approximated by a forward start at the money forward call option price. Regarding the upper bound, it can be represented as a function of european option prices on S&P500. Finally, the main advantage, in the authors point of view was that both lower and upper bounds are observables.
Zhang and Zhu [6] proposed an expression for VIX futures, assuming the S&P500 is described by Heston's stochastic volatility model. The authors presented a stochastic variance model of VIX time evolution, and developed an expression for VIX futures. Then, free parameters are estimated from market data over the past few years. From that study, it is found that the model with parameters estimated from the whole period from 1990 to 2005 overpriced the futures contracts by 16–44%, while the discrepancy is dramatically reduced to 2–12% if the parameters are estimated from the 2006-2010 year period.
Pursuant the necessity of employing a model, this study aims to propose an approximation pricing formula to evaluate the VIX futures in the double-mean reverting J. Gatheral [7, 8, 9] stochastic volatility process.
Finally, the note provides the calibration results of VIX futures under the Gatheral and the well-known Heston model.

1 Jim Gatheral's Double Mean-Reverting Stochastic Volatility Model

Our model for the S&P500 in this paper incorporates stochastic volatility in both the asset price and the volatility process. The model which was used by Gatheral for the component of the variance of the S&P500 prices, satisfies the diffusion:

$$\begin{cases} \frac{dS_t}{S_t} = rdt + \sqrt{V_t}dW_{1,t}, S_0 = s_0 > 0 \\ dV_t = \kappa\left(V_t^{'} - V_t\right)dt + \zeta_1 V_t^{\alpha} dW_{2,t}, V_0 > 0 \\ dV_t^{'} = c\left(z - V_t^{'}\right)dt + \zeta_2 {V_t^{'}}^{\beta} dW_{3,t}, V_0^{'} > 0 \\ d < W_1, W_2 >_t = \gamma dt \\ d < W_2, W_3 >_t = \rho dt \end{cases} \quad (1)$$

where:

- r_t is the constant spot interest rate,
- $V^{'}(t)$ and z are the long-run mean of $V(t)$ and $V^{'}(t)$,
- κ and c are the mean-reversion parameters because the higher there are, the more quickly the variance revert to their long-run mean,
- ζ_1 and ζ_2 are the volatility of the variance $V(t)$ and $V^{'}(t)$,
- $(W_{1,t}, W_{2,t})$ and $(W_{2,t}, W_{3,t})$ are the Brownian motions under the real measure,

It's well-known that the $S\&P500$ index and the VIX are negatively correlated while the correlations are not influence the VIX futures. From that observation, the both correlations are set to zero within this paper.
The model imposes the following constraintes:

$$\begin{cases} 2\kappa > \zeta_1^2 \\ 2c > \zeta_2^2 \\ \kappa > 0 \\ c, z, \zeta_2 \geq 0 \\ \alpha, \beta \in \left[1/2, 1\right] \end{cases} \quad (2)$$

The model follows three classification depending on the α and β values:

- we will call the case $\alpha = \beta = 1/2$ Double Heston,
- the case $\alpha = \beta = 1$ Double Lognormal,
- and the general case Double CEV

Focusing within the double lognormal classification, the VIX value formula is given by:

$$\begin{cases} \left(\frac{VIX_t}{100}\right)^2 & = & a_1 V_t + a_2 V_t^{'} + a_3 z \\ a_1 & = & \frac{1-e^{-\kappa\tau}}{\kappa\tau} \\ a_2 & = & \frac{\kappa}{\kappa - c}\left[\frac{1-e^{-c\tau}}{c\tau} - \frac{1-e^{-\kappa\tau}}{\kappa\tau}\right] \\ a_3 & = & 1 - a_1 - a_2 \\ \tau = \frac{30}{365} \end{cases} \quad (3)$$

It is easy to check that the VIX does not depend on whether we choose double Heston or lognormal or CEV.
At the end, Gatheral's model is a best candidate as it generates a term structure of the volatility skew that looks more like the observed $\frac{1}{\sqrt{T}}$ while the decay follows the well-known power law that one observed in return data.

2 VIX Futures Under Gatheral's Double Mean-Reverting Model

Carr and Wu [5] showed that, under the assumption of no arbitrage and continuous marking to market, the price of VIX futures ia a martingale under the risk-neutral probability measure $\mathbb{Q}^T$. Hence, the value of a VIX futures contract, $F(t,T)$, at time t with settlement at time T is:

$$\begin{aligned} F(t,T) &= E_t^{\mathbb{Q}^T}(VIX_T) \\ &= 100 * \int_0^{\infty} VIX_T p(t,T,VIX_T) dVIX_T \\ &= 100 * \int_0^{\infty}\int_0^{\infty} \sqrt{a_1 V_T + a_2 V_T' + a_3 z} \times \\ &\qquad f(V_T', V_T | V_t', V_t) dV_T' dV_T \end{aligned} \quad (4)$$

Clearly, to price the VIX futures, we need to find the conditional probability density function $f(V_T', V_T | V_t', V_t)$. Under the instantaneous variance process following the stochastic differential equation 1, the correspoding risk-neutral probability density is hard to determine.

We proposed to price the VIX futures by employing the convexity correction approximation. In fact, the convexity is the second order Taylor expansion, so that the expectation of equation 4 gives an approximation formula for the VIX futures in the form:

$$F(t,T) \approx \sqrt{E_t^{\mathbb{Q}^T}\left(VIX_T^2\right)} - \frac{Var_t^{\mathbb{Q}^T}\left(VIX_T^2\right)}{8\left[E_t^{\mathbb{Q}^T}\left(VIX_T^2\right)\right]^{3/2}} \quad (5)$$

where $\frac{Var_t^{\mathbb{Q}^T}\left(VIX_T^2\right)}{8\left[E_t^{\mathbb{Q}^T}\left(VIX_T^2\right)\right]^{3/2}}$ is the convexity adjustment relevant to the VIX futures.

For a more details about that approximation, we refer the readers to the appendix 3.1

In order to find out a closed-form formula for the approximate price of the VIX contract, we must find the $E_t^{\mathbb{Q}^T}\left(VIX_T^2\right)$ and $Var_t^{\mathbb{Q}^T}\left(VIX_T^2\right)$ expressions. For that purpose, let's denoted:

$$\begin{cases} var_t^{\mathbb{Q}^T}\left(VIX_T^2\right) = 100^4\left[a_1^2 var_t^{\mathbb{Q}^T}(V_T) + a_2^2 var_t^{\mathbb{Q}^T}\left(V_T'\right) + \right. \\ \left. 2a_1a_2 Cov\left(V_T, V_T'\right)\right] \\ var_t^{\mathbb{Q}^T}(V_T) = E_t^{\mathbb{Q}^T}\left(V_T^2\right) - \left[E_t^{\mathbb{Q}^T}(V_T)\right]^2 \\ var_t^{\mathbb{Q}^T}\left(V_T'\right) = E_t^{\mathbb{Q}^T}\left(V_T'^2\right) - \left[E_t^{\mathbb{Q}^T}\left(V_T'\right)\right]^2 \\ Cov(V_T, V_T') = E_t^{\mathbb{Q}^T}\left(V_T' V_T\right) - E_t^{\mathbb{Q}^T}\left(V_T'\right) E_t^{\mathbb{Q}^T}(V_T) \end{cases} \quad (6)$$

By applying Itô lemma and solving an ordinary differential equation [10], we retrieved:

$$E_t^{\mathbb{Q}^T}\left[V_T'\right] = \left(V_t' - z\right) e^{-c(T-t)} + z \quad \forall \quad T > 0 \quad (7)$$

$$\begin{aligned} E_t^{\mathbb{Q}^T}[V_T] = \frac{1}{\kappa - c}&\left[(V_t - z)(\kappa - c)\exp^{-\kappa(T-t)} + \right. \\ &\left.\kappa\left(V_t' - z\right)\left(e^{-c(T-t)} - e^{-\kappa(T-t)}\right)\right] + z \end{aligned} \quad (8)$$

$$\begin{aligned} E_t^{\mathbb{Q}^T}\left[V_T'^2\right] = e^{-\left(2c-\zeta_2^2\right)(T-t)}&\left[V_t'^2 - \frac{2cz\left(V_t' - z\right)}{c - \zeta_2^2} - \frac{2cz^2}{2c - \zeta_2^2}\right] + \\ &\frac{2cz\left(V_t' - z\right)}{c - \zeta_2^2} e^{-c(T-t)} + \frac{2cz^2}{2c - \zeta_2^2} \end{aligned} \quad (9)$$

$$\begin{cases} E_t^{\mathbb{Q}^T}\left[V_T V_T'\right] = \left(V_t V_t' - \sum_{i=1}^{4} a_{ii}\right) e^{-(\kappa+c)(T-t)} + a_{11} e^{-\left(2c-\zeta_2^2\right)(T-t)} + \\ a_{22} e^{-c(T-t)} + a_{33} e^{-\kappa(T-t)} + a_{44} \\ a_{11} = \frac{\kappa\left[V_t'^2\left(c-\zeta_2^2\right)\left(2c-\zeta_2^2\right) - 2cz\left(V_t'\left(2c-\zeta_2^2\right) - cz\right)\right]}{\left(c-\zeta_2^2\right)\left(2c-\zeta_2^2\right)\left(\kappa-c+\zeta_2^2\right)} \\ a_{22} = cz\frac{\left(V_t' - z\right)\left(2\kappa - c - \zeta_2^2\right)}{\left(c-\zeta_2^2\right)(\kappa - c)} \\ a_{33} = \frac{z\left[\left(V_t - V_t'\right)\kappa + c(z - V_t)\right]}{(\kappa - c')} \\ a_{44} = \frac{cz^2\left[2\kappa + 2c - \zeta_2^2\right]}{(\kappa + c)\left(2c - \zeta_2^2\right)} \end{cases} \quad (10)$$

$$\begin{cases} E_t^{\mathbb{Q}^T}\left[V_T^2\right] = \left(V_t^2 - \sum_{i=0}^{4} b_{ii}\right) e^{-\left(2\kappa-\zeta_1^2\right)(T-t)} + b_{00} e^{-(\kappa+c)(T-t)} + \\ b_{11} e^{-\left(2c-\zeta_2^2\right)(T-t)} + b_{22} e^{-c(T-t)} + b_{33} e^{-\kappa(T-t)} + b_{44} \\ b_{00} = \frac{2\kappa\left(V_t V_t' - \sum_{i=1}^{4} a_{ii}\right)}{\kappa - c - \zeta_1^2} \\ b_{11} = \frac{2\kappa a_{11}}{2\kappa - 2c + \zeta_2^2 - \zeta_1^2} \\ b_{22} = \frac{2\kappa a_{22}}{2\kappa - c - \zeta_1^2} \\ b_{33} = \frac{2\kappa a_{33}}{\kappa - \zeta_1^2} \\ b_{44} = \frac{2\kappa a_{44}}{2\kappa - \zeta_1^2} \end{cases} \quad (11)$$

It's straightforward that these expectations have finite value because of the model's parameters conditions disclosed within formula 2.

By substituting the formulas 7, 8, 9, 11 and 10 into 6, one obtain the value of $E_t^{\mathbb{Q}^T}\left(VIX_T^2\right)$ and $Var_t^{\mathbb{Q}^T}\left(VIX_T^2\right)$ and recursevely the value of the VIX futures.

We have compared the calibration of the VIX futures (provided by equation 5) under the 2007-2008 financial crisis market data and results are provided at figure 1.

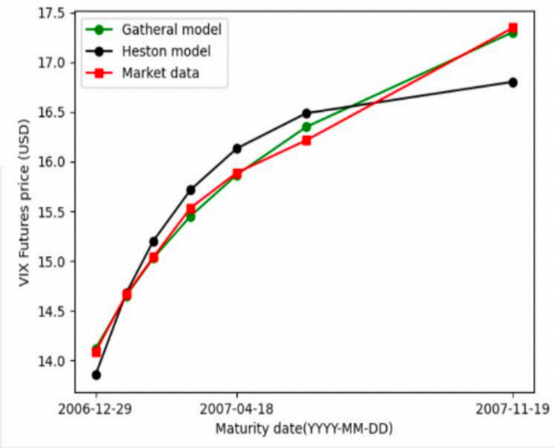

Figure 1: VIX futures calibration retrieved from the approximated closed-formula for the period between 2006 of december and november 2007.

There is a clear evidence that Gatheral's double lognormal model outputs are closed to the market data than those of Heston (More details about the VIX and VIX futures closed-formula under Heston model is provided in appendix 3.2) model.

Conclusion

We highlighted the Gatheral's double mean-reverting model and their main features. Then, we used the convexity to propose an approximated closed-formula for VIX futures. Even if the calibration output of Gatheral's double lognormal model is of interesting, it is important to mention that the number of free parameters needed to access that accuarcy is relatively high. In fact, while Gatheral's double lognormal model uses seven (under non correlation assumption) parameters, Heston uses four parameters.
The paper is of interesting as the proposed formula does not refer to an integral, but is a much shorter computational time needed to compute the price of a VIX futures contract in comparison with the Monte Carlo simulation. Finally, the determination of model parameters should also greatly facilitated. Follow-up paper will focus on the calibration of the Capped Volatility Swaps under the CEV Gatheral's model.

3 Appendix

3.1 Taylor development

This section aims to provide details for the reader
Let's considered a function f of variable x, at least twice differentiable. The Taylor development, arround a point x_0 of function f is:

$$f(x) = f(x_0) + (x - x_0) f^{'}(x_0) + \frac{1}{2}(x - x_0)^2 f^{''}(x_0) + o(x - x_0)^3 \tag{12}$$

Define:

$$\begin{cases} f(X) = \sqrt{X} \\ X_0 = E_t^{\mathbb{Q}^T}(X) \end{cases} \tag{13}$$

Based on 20, equation 12 becomes:

$$f(x) \approx \sqrt{E_t^{\mathbb{Q}^T} X} + \frac{1}{2}\left(x - E_t^{\mathbb{Q}^T} X\right)\left(E_t^{\mathbb{Q}^T} X\right)^{-\frac{1}{2}} - \frac{1}{8}\left(x - E_t^{\mathbb{Q}^T} X\right)^2 \left(E_t^{\mathbb{Q}^T} X\right)^{-\frac{3}{2}} \tag{14}$$

Taking the expectation of the above equation yields:

$$E_t^{\mathbb{Q}^T}[f(x)] \approx \sqrt{E_t^{\mathbb{Q}^T} X} - \frac{1}{8} E_t^{\mathbb{Q}^T}\left[\left(x - E_t^{\mathbb{Q}^T} X\right)^2\right]\left(E_t^{\mathbb{Q}^T} X\right)^{-\frac{3}{2}} \tag{15}$$

Replacing X by VIX_T^2 gives:

$$E_t^{\mathbb{Q}^T}[VIX_T] \approx \sqrt{E_t^{\mathbb{Q}^T}(VIX_T^2)} - \frac{1}{8} var_t^{\mathbb{Q}^T}(VIX_T^2)\left[E_t^{\mathbb{Q}^T}(VIX_T^2)\right]^{-\frac{3}{2}} \tag{16}$$

$$E_t^{\mathbb{Q}^T}[(VIX_T)] \approx \sqrt{E_t^{\mathbb{Q}^T}(VIX_T^2)} - \frac{1}{8}\frac{var_t^{\mathbb{Q}^T}(VIX_T^2)}{\left[E_t^{\mathbb{Q}^T}(VIX_T^2)\right]^{\frac{3}{2}}} \tag{17}$$

3.2 VIX Futures under Heston model

In the Heston model[11, 12], the stochastic variance follows the below CIR dynamics:

$$\begin{cases} \frac{dS_t}{S_t} = r dt + \sqrt{V_t} dW_{1,t}, S_0 = s_0 > 0 \\ dV_t = \kappa(\theta - V_t) dt + \zeta V_t^{\alpha} dW_{2,t}, V_0 = v_0 > 0 \\ d < W_1, W_2 >_t = \rho dt \end{cases} \tag{18}$$

where $(W_{1,t}, W_{2,t})$ are correlated Browian motion under the risk neutral measure. The free parameters κ, θ and ζ are respectively the speed of mean reversion, the long term mean level and the volatility of the variance, while r and S_t are the risk free rate and the $S\&P500$ index spot value.
The VIX formula for this model yields:

$$\begin{cases} \left(\frac{VIX_t}{100}\right)^2 = aV_t + b \\ a = \frac{1 - e^{-\kappa\tau}}{\kappa\tau} \\ b = \theta(1 - a) \\ \tau = \frac{30}{365} \end{cases} \tag{19}$$

Finally, the VIX futures formula is given by:

$$\begin{cases} F(t,T) = 100 \int_0^{+\infty} \sqrt{aV_T + b} \times f_{T-t}(V_T|V_t)\, dV_T \\ f_{T-t}(V_T|V_t) = C \times e^{-u-v} \left(\frac{v}{u}\right)^{\frac{q}{2}} I_q\left(2\sqrt{uv}\right) \\ C = \frac{2\kappa}{\sigma^2\left(1-e^{-\kappa(T-t)}\right)} \\ u = CV_t e^{\kappa(T-t)} \\ v = CV_T \\ q = \frac{2\kappa\theta}{\sigma^2} - 1 \end{cases} \tag{20}$$

where $I_q(z)$ is the modified Bessel function given by:

$$I_q(z) = \left(\frac{1}{2}z\right)^q \sum_{k=0}^{+\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(q+k+1)} \tag{21}$$

and $\Gamma(q+k+1)$ is the gamma function for real number $q+k+1$

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A propos d'Awalee

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