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AWALEE NOTES

A UNIFIED PDE MODELLING FOR CVA AND FVA

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AWALEE PRESENTATION

Chapter 0

INTRODUCTION

The recent finance crisis has released the counterparty risk in the valorization of the derivatives. The difference between the evaluations before and after crisis is computed as x value adjustment (xVA). Due to the possibility of the counterparty's default, the credit value adjustment (CVA) is firstly introduced into xVA. In order to cover this risk, the tendency is the trading with collateral. The cost of the collateral is referenced as the liquidity value adjustment (LVA) [1]in xVA. Another consequence of the crisis is the absence of a unified funding rate (risk free). The refunding rate of the bank causes the funding valuation adjustment (FVA) as the third component of xVA. Consequently, we need to integrate the xVA (CVA+FVA+LVA) into the classic pricing PDE. In general, the counterparty's default can be modeled with three levels (independent, immersion and beyond immersion) of the dependency on the underlying price. In the chapter 1 and 2, we introduce briefly the general modelling of the default time and xVA modelling on a portfolio under the immersion case. The annex gives more details on the adequateness of the immersion case. In the last chapter, we give some numeric results under the dynamic Gaussian copula model.

Chapter 1

F/G FILTRATIONS

We consider a default free filtration $\mathbb{F}$ generated by a Brownian motion. The filtration $\mathbb{G}$ is enlarged progressively from $\mathbb{F}$ by a default time τ with the indicator process $H_t = 1_{\{\tau\leq t\}}$ that is, $G_t = F_t \vee \sigma(\vee_{s\leq t} H_s)$. We assume that it exists a pricing measure $\mathbb{Q}$ on the filtration $\mathbb{G}$.

In the filtration $\mathbb{F}$, we suppose that the dynamics of the underlying price follows

$$dX_t/X_t = r_t\,dt + \sigma dB_t \qquad \text{in } \mathbb{F} \qquad (1)$$

where B_t is the $(\mathbb{F},\mathbb{Q})$-Brownian motion. From the no-arbitrage condition, the process r_t in (1) should be the repo rate of the underlying under the pricing measure $\mathbb{Q}$. Under the hypothesis that the OIS rate is equal to the repo rate as before crisis, we denote the OIS rate by $\mathbf{r}$ to represent the repo rate.

[1] It is also referenced as the collateral rate adjustment for some practitioner.

In the default-free period (before crisis), any derivative product with payoff/claim $\Psi(X_T)$ at maturity T has the fair value at time t

$$M_t = E[\,e^{-\int_t^T r_u\,du}\,\Psi(X_T)\,|\,X_t] \qquad (2)$$

, with X_t defined in (1). Actually, it is the probabilistic solution of the following classic Black-Sholes EDP,

$$\partial_t M + \frac{1}{2}\sigma^2 x^2\,\partial_{xx}M + r_t\,x\,\partial_x M = r_t\,M, \qquad (3)$$
$$M(T,.)=\Psi(X_T).$$

In the filtration $\mathbb{G}$, the behavior of the $(\mathbb{F},\mathbb{Q})$-Brownian motion B_t will be changed under the influence of the information on the default time. Here, we assume

- The $(\mathbb{F},\mathbb{Q})$ -Brownian motion B_t remains a semi-martingale with the canonical decomposition

$$dB_t = dW_t + \Delta_t dt \qquad \text{in } G$$

, where W_t is a $(\mathbb{G},\mathbb{Q})$-Brownian motion. So, the dynamics can be written in

$$\begin{aligned} dX_t/X_t &= r_t dt + \sigma(dW_t + \Delta_t dt) \\ &\triangleq \gamma_t dt + \sigma dW_t. \end{aligned} \qquad \text{in } G$$

- The default time τ admits a $\mathbb{G}$–predictable intensity. It can be decomposed in two part: an $\mathbb{F}$–predictable process λ_t (pre-default) before the time of the default and 0 after the default.

Under the two previous assumptions, the relationship between the filtration $\mathbb{F}$ and the filtration $\mathbb{G}$ can be defined according to the three different cases

- Independent case, that is, the pre-default intensity process λ_t is deterministic and $\gamma_t \equiv r_t$.
- Immersion case, that is, $\gamma_t \equiv r_t$.
- Beyond immersion case, that is λ_t, is a function of (t, B_t) and $\gamma_t \neq r_t$.

For the non independent case, the impact bilateral between τ and B_t is given respectively by λ_t (no deterministic) and Δ_t. Under the pricing measure $\mathbb{Q}$, the choice of the immersion case, that is $\Delta_t \equiv 0$, will be justified in the next chapter and the credit risk/counterparty risk quant focus on the modeling of the λ_t to treat the dependence between τ and B_t.

Chapter 2

XVA(CVA/LVA/FVA) – PDE MODELING

1 CVA PART

Let τ be the default time of the counterparty on the derivative product with the default free fair value M_t defined in (2). The t-time value U_t is the counterparty risky fair value for the bank. The no-arbitrage condition and the completeness of the market [2]give the **(G, Q)** -martingale condition on $U(t, X_t)$, that is,

$$\partial_t U + \frac{1}{2}\sigma^2 x^2 \partial_{xx} U + \gamma_t \, x \, U + \lambda_t(\bar{U}_t - U) = r_t U \qquad (4)$$
$$U(T,.) = \Psi(X_T)$$

where $\bar{U}_t$ is the derivative value just after the default of the counterparty. For the no collateral case, $\bar{U}_t$ is given by[3]

$$\bar{U}_t = R\, {M_t}^+ - {M_t}^-$$

with **R** the recovery rate (assumed constant).

In the following part, the bank's funding rate and the collateral rate will be involved in the fair value U_t.

2 FVA/LVA PART

A CSA (cash) collateralization schemas is used between the bank and the counterparty in order to mitigate CVA. We denote the collateral value posted by the bank before the default time τ by the process Γ_t and the xVA-PDE (5) can be derived [4]from (4) with $\bar{U}_t = R\,(M_t - \Gamma_t)^+ - (M_t - \Gamma_t)^- + \Gamma_t$ after the default and U_t before the default

$$\partial_t U + \frac{1}{2}\sigma^2 x^2 \partial_{xx} U + \gamma_t x \partial_x U + \lambda_t \; [\, R\,(M_t - \Gamma_t)^+ - (M_t - \Gamma_t)^- - (U_t - \Gamma_t)\,]$$
$$= (r_t + r_t^F)\,(U_t - \Gamma_t) + (r_t + r_t^C)\,\Gamma_t$$

$$\partial_t M + \frac{1}{2}\sigma^2\, x^2\, \partial_{xx} M + r_t\, x\, \partial_x M = r_t M,$$

$$M(T,.) = U(T,.) = \Psi(X_T). \qquad (5)$$

In the right of (5), the OIS rate r_t is used as a reference for all the other funding rates. The $r_t^{\,C}$ and $r_t^{\,F}$ are the extra remuneration of the collateral and the refunding rate of the bank related to r_t.

The xVA value V_t is given by $V_t = U_t - M_t$, where U_t and M_t are respectively the solution of the linear EDP in (3) and (5).

We remark that if $\gamma_t \equiv r_t$ holds in (5), the xVA-EDP on V_t will be simplified substantially. Actually, this is the case if we consider the underlying can be refunded by the repo market.

[2] The left part in (4) is derived from Ito's formulas and the right part is the **(G-Q)**-increment rate of the auto financed porfolio.
[3] $x = x^+ - x^-$
[4] The right part in (5) is the **(G-Q)** -increment rate of the auto financed porfolio in our finding environement ***(See Vladimir Piterbarg)***.

3 THE CHOICE OF γ_t

Recall that under the pricing measure **Q**, the drift part γ_t in the dynamics of X_t should be the repo rate. Furthermore, we assumed the repo rate is always close to the OIS rate, that is,

$$dX_t/X_t = r_t\, dt + \sigma dB_t \qquad \text{in } G$$

or equivalently $\gamma_t \equiv r_t$. In this case, the XVA-PDE is given by

$$\partial_t V + \frac{1}{2}\sigma^2 x^2 \partial_{xx} V + r_t x \partial_x V = r_t V_t \; +$$

$\lambda_t\,(-R)(M_t - \Gamma_t)^+ + \lambda_t V_t$	+	$r_t^F\,(V_t + M_t - \Gamma_t)$	+	$r_t^C\,\Gamma_t$
CVA		FVA		LVA

$$V(T,.) = 0 \qquad (6)$$

This is a linear PDE, thus the probabilistic solution is

$$V_t = E[\int_t^T -[\lambda_s\,(1-R)(M_s - \Gamma_s)^+ + r_s^F M_s + (r_s^C - r_s^F)\Gamma_s]\; e^{\int_t^s -(r_u^F + r_u + \lambda_u)du}\, ds \;|W_t] \qquad (7)$$

, where M_t, Γ_t, and λ_t are the function of the (t, W_t) with $dX_t / X_t = r_t dt + \sigma dW_t$.

For the independent case, the immersion case for $\gamma_t \equiv r_t$ holds obviously. For the beyond immersion case, we show the existence of the pricing measure such that the process λ_t does not change and $\gamma_t \equiv r_t$ holds in Annex. Consequently, the immersion case is adequate for xVA modelling as the dependence is expressed in terms of λ_t.

In the next chapter, we will give the numeric results associated to xVA value computed on the dynamized Gaussian copula default time model (DGC model) under the pricing measure.

Chapter 3

NUMERICS :

In this chapter, we use the dynamized Gaussian copula model to model a default time and τ an underlying price process X_t. They are respectively generated by a bi-variate Brownian motion $B = (B^1, B^2)$ with a constant correlation ρ.

- $\tau = h^{-1}(\int_0^{\infty} f(s) dB_s^1)$
- $dX_t/X_t = r_t\, dt + \sigma dB_t^2$

The details of DGC model is given in Annex. The call option on the underlying can be evaluated by the Black-Sholes formulas before crisis, for example $M_0 = 1.0266$, for $X_0 = 2$, $Strike = 1$, $r = 0$ $\sigma = 0.5$ $Matury = 1$. We will show the xVA value on this call option.

The Monte-Carlo simulation of (7) needs:

- The markovian process in (7) is a bi-variate Brownian motion $B = (B^1, B^2)$ in DGC model.
- The pre-default intensity process λ_t is in function of (t, B_t^1) given in (10).
- The underlying price M_t is given by the Black-Sholes formulas.

The collateral Γ_t is taken by two case $\Gamma_t = M_t$ and $\Gamma_t = 0$ and the OIS, the collateral and the funding rate are constant in this numeric test.

- $\Gamma_t = 0$

With our funding environment $r = r_f = r_c = 0$ and recovery of counter-party risk is assumed at 40%, we have the following CVA result of

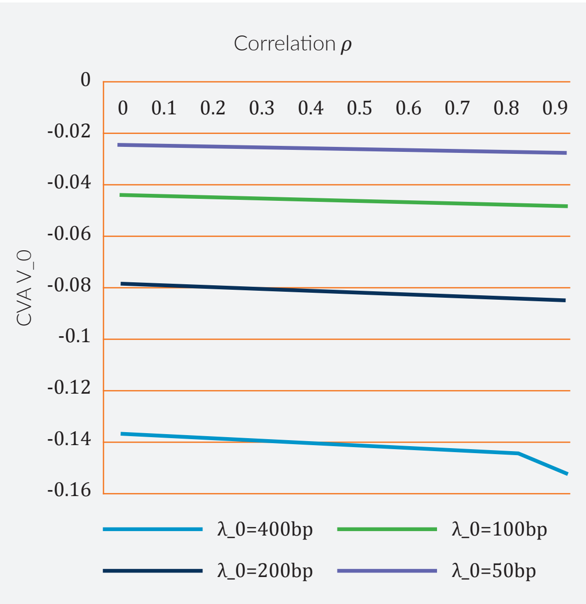


Figure (8): CVA versus correlation parameter ρ in DGC (Monte Carlo with 10000 scenarios)

$$V_0 = E\left[\int_0^T -\lambda_s (1-R) M_s^{+}\ e^{\int_t^s -\lambda_u du}\, ds\right]$$

λ_0 is given as a DGC model parameter for the level of the intensity of τ. The CVA value is determined by the parameter λ_0 and the correlation ρ between the underlying price (exposure) and the default risk as show in figure (8).

- $\Gamma_t = M_t$

With $r = 0$ and, we have the following LVA result of

$$V_0 = E[\int_0^T -r_c M_s e^{-\int_0^s (r_f + \lambda_u) du}\, ds].$$

$\lambda_0 = 200bp$			
	$r_c = 200bp$	$r_c = 400bp$	$r_c = 600bp$
$r_f = 0$	−0.0188	−0.0376	−0.0565
$r_f = 200bp$	−0.0186	−0.0373	−0.0559
$r_f = 400bp$	−0.0184	−0.0369	−0.0554
$r_f = 600bp$	−0.0183	−0.0366	−0.0549

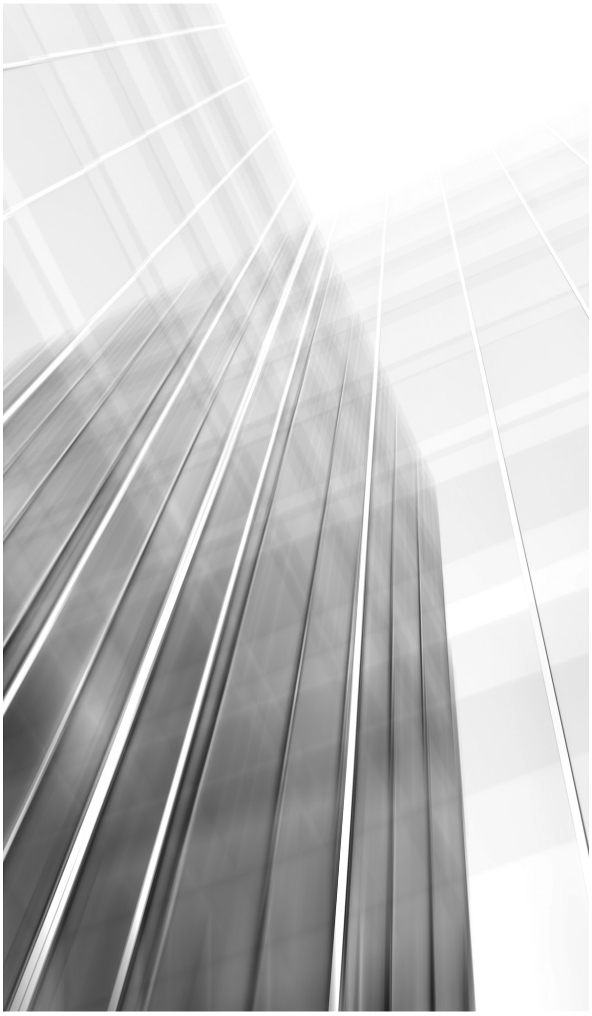
$\lambda_0 = 400bp$			
	$r_c = 200bp$	$r_c = 400bp$	$r_c = 600bp$
$r_f = 0$	−0.0177	−0.0355	−0.0532
$r_f = 200bp$	−0.0175	−0.0351	−0.0527
$r_f = 400bp$	−0.0174	−0.0348	−0.0522
$r_f = 600bp$	−0.0172	−0.0345	−0.0517

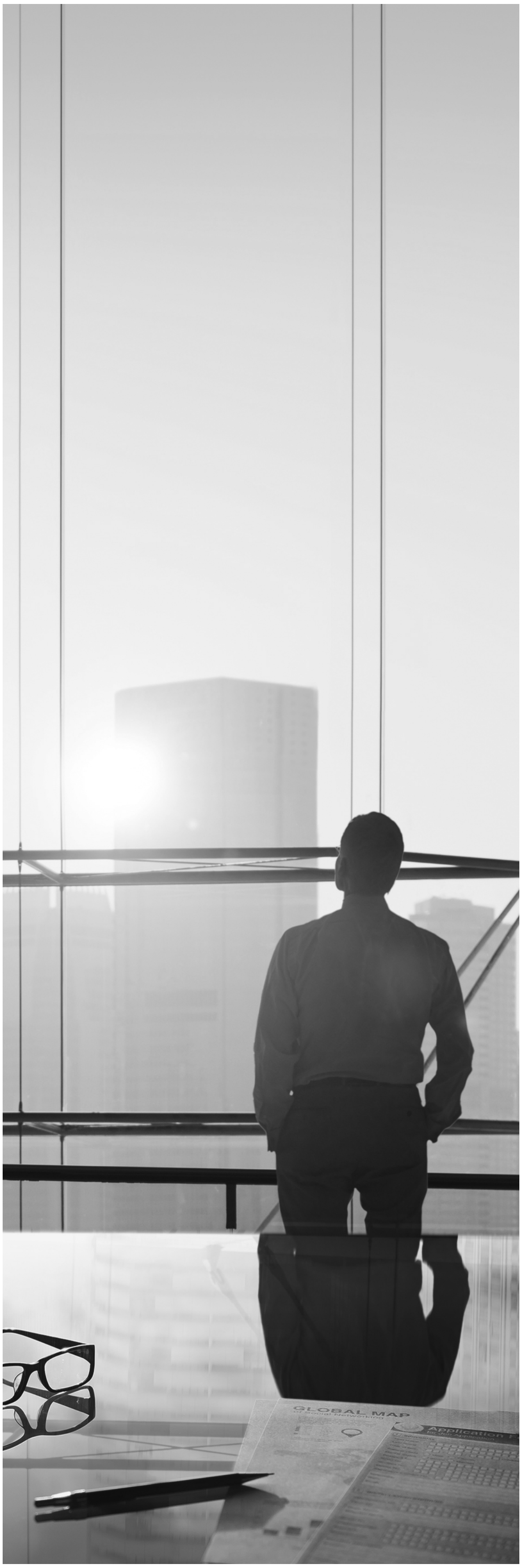
Tables (9): LVA in fully collateral case (Monte Carlo with 10000 scenarios)

The xVA value V_0 is dominated by the collateral rate r_c . The funding rate r_f is remuneration of the part $U - M$, which has the minuscule impact. The effective maturity $T \wedge \tau$ decreases with the increasement of the intensity of the default time, the absolute value of xVA value will also decrease.

CONCLUSION

In this note, we have shown that the modelling of the counter-party's default time involves three levels of dependence with the underlying price. Under the pricing measure chosen by the repo market, the immersion case is suitable for xVA PDE. The dependence is only modeled by the intensity process. In particular, the correlation parameter ρ of the DGC model plays an important role to treat the wrong way risk on CVA. Actually, the case beyond immersion focus on what happens after the default time see N. El Karoui, so that it is suitable to deal with credit portfolio or gap risk. For the fully collateral trading, we can see the xVA is dominated by the LVA part while the CVA and FVA parts have little impact on the xVA value.





ANNEX

A: Density framework01

As the default time τ is a positive random variable, it is fully characterized by its survival function, that is, $P(\tau > t)$, or equivalently to the density function $P(\tau \epsilon dt)$, when it exists. In order to establish the relationship between τ and filtration $\mathbb{F}$, we use the $\mathbb{F}$-conditional survival process $F_t(v) \triangleq P(\tau > v \mid F_t)$ or the conditional density process $f_t(v) \triangleq P(\tau \epsilon dv \mid F_t)$ to model the default time.

The intensity process of the default time can be deduced by the density process (see (10)). Consequently, the density modeling of the default time is a framework more general than our assumption in the first part.

The pricing filtration $\mathbb{G}$ is produced by the filtration $\mathbb{F}$ progressively enlarged by the default time τ, so for every t,

$$G_t = F_t \vee \sigma(\vee_{s \le t} H_s)$$

The predictable representation theorem in the filtration $\mathbb{G}$ is generated by two fundamental martingales:

- 1. The $(\mathbb{G}, \mathbb{P})$-Brownian motion W_t defined

$$B_t = W_t + \int_0^{\tau \wedge t} \frac{d < B, F >_u}{F_u(u)} + \left(\int_{t \wedge v}^{t} \frac{d < B, f(v) >_u}{f_u(v)}\right)_{|\, v=\tau}$$

$$\triangleq W_t + \int_0^t \Delta_s ds$$

- 2. The compensated martingale N_t defined as

$$dN_t = dH_t - \lambda_t dt$$

, with the pre-default intensity of τ

$$\lambda_t = \frac{f_t(t)}{F_t(t)} \qquad (10)$$

So, any (G, P)- Radon-Nikodým density L_t can be written as a Doléans-Dade exponential driven by the $(\mathbb{G}, \mathbb{P})$-Brownian motion W_t and the compensated martingale N_t

$$L_t = 1 + \int_0^t L_{s-}\, \alpha_s dW_s + \int_0^t L_{s-}\, \beta_s dN_s$$

where the processes α_t and β_t are $\mathbb{G}$–predictable.

In particular, the $(\mathbb{G}, \mathbb{P})$-Radon-Nikodým density $L_t = 1 - \int_0^t L_{s-} \Delta_s dW_s$ defines a new probability measure $\mathbb{Q}$ on $\mathbb{G}$ such that

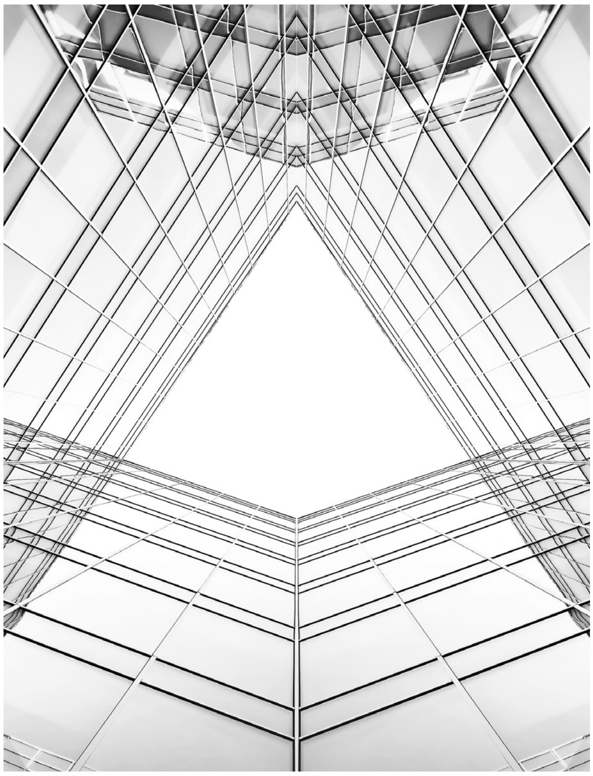
$$W_t = \bar{W}_t - \int_0^t \Delta_s ds$$

, where $\bar{W}_t$ is the $(\mathbb{G}, \mathbb{Q})$-Brownian motion. Furthermore, the $(\mathbb{F}, \mathbb{P})$-Brownian motion B_t remains the $(\mathbb{F}, \mathbb{Q})$-Brownian motion, as the projection of $(\mathbb{G}, \mathbb{P})$-Radon-Nikodým density L_t on the filtration $\mathbb{F}$ is 1, that is $E[Lt|Ft] \equiv 1$ (See dongli wu).

Consequently, the $(\mathbb{G},\mathbb{Q})$-Brownian motion $\bar{W}_t$ given by

$$\bar{W}_t = \bar{W}_t^i - \int_0^t \Delta_s ds + \int_0^t \Delta_s ds = W_t + \int_0^t \Delta_s ds = B_t$$

remains a $(\mathbb{F},\mathbb{Q})$-Brownian motion B_t, so the immersion holds between $\mathbb{F}$ and $\mathbb{G}$ under the pricing measure $\mathbb{Q}$. Furthermore, the compensated martingale N_t does not change as it is orthogonal of the W_t, so the intensity does not change motion under the new probability measure $\mathbb{Q}$.



B: A beyond immersion case with dynamized Gaussian copula model

One considers a bi-variate Brownian motion $B = (B^1, B^2)$ with pairwise correlation ρ in its own completed filtration $\mathbb{F}$ under a probability measure $\mathbb{P}$. Let h be a differentiable increasing function from R^+ to R with $lim_{s\to 0} h(s) = -\infty$ and $lim_{s\to\infty} h(s) = +\infty$. We defined the default time

$$\tau = h^{-1}(\int_0^\infty f(s) dB_s^1)$$

, with f is a square integrable function with unit L^2-norm. In our numeric test, we take $f \equiv 1$ and the horizon ∞ by 5 years. The time horizon plays a role of the volatility of the intensity.

We denote $\sigma^2(t) = \int_t^\infty f^2(s) ds$ and $m_t^i = \int_0^t f(s)\, dB_t^i$, so

$$\{\tau > t\} = \left\{ \int_t^\infty f(s) dB_s^1 > h(t) - m_t^1 \right\}$$

The $\mathbb{F}$-conditional survival process of τ is given as

$$F_t(v) = P(\tau > v \mid F_t) = \Phi(\frac{h(v) - m_t^1}{\sigma(t)})$$

, with Φ is the standard Gaussian survival function.

The $(\mathbb{F}, \mathbb{P})$-density process of τ is given as

$$f_t(v) = P(\tau \in dv \mid F_t) = \phi(\frac{h(v) - m_t^1}{\sigma(t)}) \frac{h'(v)}{\sigma(t)}$$

Let F be the Azéma supermartingale of τ, that is, $F_t = P(\tau > t \mid F_t)$. The dynamics of $f(v)$ and F are given by

$$df_t(v) = f_t(v) \alpha_t(v) dB_t \quad dF_t = f_t(t) dt + \beta_t dB_t$$

with

$$\alpha_t(v) = \frac{h(v) - m_t^1}{\sigma(t)} \frac{f(t)}{\sigma(t)}$$

$$\beta_t = \phi\left(\frac{m_t^1 - h(t)}{\sigma(t)}\right) \frac{f(t)}{\sigma(t)}$$

The fundamental $(\mathbb{G}, \mathbb{P})$-martingales:

The $(\mathbb{F}, \mathbb{P})$-Brownian motions $(\mathrm{B1}, \mathrm{B2})$ remain the semi-martingale with the following $\mathbb{G}$-canonical decomposition of the B, for i=1,2,

$$B_t^i = W_t^i + \int_0^{\tau\wedge t} \frac{d < B^i, F >_u}{F_u} + (\int_{t\wedge v}^t \frac{d < B^i, f(v) >_u}{f_u(v)})_{|\, v=\tau}$$

$$\triangleq W_t^i + \int_0^t \Delta_s^i ds$$

, where (W^1, W^2) are $(\mathbb{G}, \mathbb{P})$-Brownian motions. Precisely, we have

$$\Delta_t^1 = \int_0^{\tau\wedge t} \frac{\beta_u}{F_u} du + (\int_{t\wedge v}^t \alpha_u(v) du)_{|\, v=\tau}$$

$$\Delta_t^2 = \int_0^{\tau\wedge t} \frac{\beta_u}{F_u} \rho du + (\int_{t\wedge v}^t \alpha_u(v) \rho du)_{|\, v=\tau}$$

and the compensated martingale

$$dN_t = dH_t - \lambda_t dt$$

, with the pre-default intensity of τ

$$\lambda_t = \frac{f_t(t)}{F_t}$$

The pricing measure $\mathbb{Q}$ such that the immersion property holds:

Let $L_t = 1 - \int_0^t L_{s-} \Delta_s^1 dW_t^1$ be a $(\mathbb{G}, \mathbb{P})$-Radon-Nikodým densitydefining a new probability measure $\mathbb{Q}$. From Girsanov' theorem, we can verify the

$$W_t^i = \bar{W}_t^i - \int_0^t \Delta_s^i ds$$

,where $\bar{W}_t^i$ are the $(\mathbb{G}, \mathbb{Q})$-Brownian motion. Consequently, the $(\mathbb{F}, \mathbb{P})$-Brownian motion (B^1, B^2) given by

$$B_t^i = W_t^i + \int_0^t \Delta_s^i ds = \bar{W}_t^i - \int_0^t \Delta_s^i ds + \int_0^t \Delta_s^i ds = \bar{W}_t^i$$

remain the Brownian motion under the probability $\mathbb{Q}$ in the filtration $\mathbb{G}$. We recall that the intensity does not change motion under the new probability measure $\mathbb{Q}$.

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