AWALEE NOTES

CVA CALCULATION UNDER WRONG WAY RISK

Focus on Rosen & Saunders model and IPFP algorithm

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1. INTRODUCTION

Since the subprime crisis, Credit Value Adjustment (CVA), which is the market value of counterparty credit risk, is one of the most important credit risk measures recognized by both industry practitioners and regulators. To calculate this price, a lot of methods are presented in the literature. An important event that can be taken into account in CVA formula is the so-called Wrong Way Risk (WWR). It is the event that « occurs when exposure to the counterparty is adversely correlated with the credit quality of that counterparty » as defined by ISDA (International Swaps and Derivatives Association) in 2001.

The objective of this paper consists in researching on Wrong Way Risk modeling. In the first section, we will recall the general formula for CVA calculation. Secondly, we will study the model proposed by Rosen and Saunders to calculate the CVA with WWR. In the last section, we study the IPFP (Iterative Proportional Fitting Procedure) algorithm and its application in the CVA calculation under Wrong Way Risk as well.

2. CVA : GENERAL FORMULA

Credit Value Adjustment (CVA) is the difference between the risk free portfolio value and the true counterparty default risky portfolio value. In other terms, it's the market value of the counterparty credit risk.

2.1. Notations

Throughout this report, we will need and use these notations :

- $V(t)$: Market-value of the contract at time t.
- R_c : Recovery rate (the fraction of the portfolio value that can be recovered in case of a default), assumed constant.
- $LGD=1-R_c$: Loss Given Default.
- X^+ : Positive part of a real X.
- $E_x(t)$: Exposure to a counterparty at time t ($E_x(t) = V(t)^+$)
- τ : Default time of the counterparty.
- F_τ : Cumulative distribution function of the random variable τ
- λ : Default intensity of the counterparty.
- T : Maturity of the contract.
- r_t : Interest rate at time t.
- $D(u,v)$: Discount factor between u and v.
- $\mathbb{E}^{\mathbb{P}}$: Expectation sign under the real-world measure $\mathbb{P}$.
- $\mathbb{E}^{\mathbb{Q}}$: Expectation sign under neutral-risk measure $\mathbb{Q}$. (Thereafter we use $\mathbb{E}$ instead of $\mathbb{E}^{\mathbb{Q}}$).
- $(F_t)_{0\leq t\leq T}$: Filtration
- $\Phi(u) = \frac{1}{\sqrt{2\pi}} exp(-u^2/2)$: *Standard gaussian density function*
- $\Psi(x) = \int_{-\infty}^{x} \Phi(u)du$: *Standard gaussian cumulative function*

2.2. Hypothesis

We will also work under the following hypothesis :

- τ follows an exponential distribution : $F_\tau(t) = 1 - exp(-\lambda t)$
- Rates are deterministic : $D(u,v) = exp(-\int_u^v r_s ds)$
- The underlying S^i follows Black-Scholes model :

$$\frac{dS_t^i}{S_t^i} = rdt + \sigma^i dW_t^i$$

W_t^i : brownian motion under mesure $\mathbb{Q}$

2.3. General formula

The CVA is priced under risk-neutral measure and the formula (at time zero) is given by :

$$\begin{aligned} CVA &= LGD\, \mathbb{E}(E_x(\tau)D(0,\tau)1_{\tau<T}) \\ &= LGD \int_0^T \mathbb{E}(D(0,u)E_x(u)|\tau = u)d\mathbb{Q}_\tau(u) \\ &= LGD \int_0^T D(0,u)\mathbb{E}(V(u)^+|\tau = u)\lambda exp(-\lambda u)du \end{aligned}$$

By discretization, we can write :

$$CVA = LGD \sum_{i=1}^{N} D(0,t_i)\mathbb{E}[V(t_i)^+|t_{i-1} \leq \tau \leq \tau_i](F_\tau(t_i) - F_\tau(t_{i-1}))$$

$$0 \leq t_0 \leq t_1 \leq ... \leq t_N$$

If we suppose exposure and default time are independent and we don't have an analytical formula for $\mathbb{E}(V(t_i^+))$, we generate M price scenarios using Monte Carlo simulation. Then the formula becomes :

$$CVA = \frac{1}{M} LGD \sum_{i=1}^{N}\sum_{j=1}^{M} D(0,t_i)\left(\frac{V_j(t_{i-1})^+ + V_j(t_i)^+}{2}\right)\mathbb{Q}(t_{i-1} \leq \tau \leq t_i) \quad (1)$$

The sum $EAD_i = \frac{1}{M}\sum_{j=1}^{M} D(0,t_i)\left(\frac{V_j(t_{i-1})^+ + V_j(t_i)^+}{2}\right)$ is known as Exposure at default at time t_i.

3. WRONG WAY RISK

3.1. Introduction

The dependency structure between counterparty default events and the portfolio value is sometimes non-negligible.

In this case we say that there is a "market-credit" dependency. When we have a positive dependency (exposure increases with the counterparty default probability), «one says» that there is Wrong Way Risk (WWR). However, when exposure decreases with the default probability we say that we have Right-Way Risk (RWR). A straightforward example of WWR is a long position in a European put.
In order to estimate WWR, a number of models have been proposed. In the literature, there is no indication as to which of these models is the best and optimal one. Rosen and Saunders model is the classic one, it will be the subject of the next paragraph.

3.2. Rosen and Saunders model

Rosen and Saunders proposed an intuitive method for modeling WWR which is robust and easy to implement. The major advantage of this method consists in using pre-computed exposures in the case of independence between exposures and counterparty credit quality. These exposures are generated by Monte Carlo simulation. A similar discretisation to equation (1) that takes into account the correlation of exposures and counterparty credit risk leads to this expression :

$$CVA = \frac{1}{M} LGD \sum_{i=1}^{N} \sum_{j=1}^{M} D(0,t_i) \left(\frac{V_j(t_{i-1})^+ + V_j(t_i)^+}{2} \right) A$$

where $A = \mathbb{Q}(\omega = \omega_j | t_{i-1} \leq \tau \leq t_i)(F_\tau(t_i) - F_\tau(t_{i-1}))$

w_j corresponds to scenario j.

We can also approximate CVA with another alternative discretization : That is to say that default occurs exactly at time t_i and not in the interval $[t_{i-1}, t_i]$. This leads to this formula :

$$CVA = \frac{1}{M} LGD \sum_{i=1}^{N} \sum_{j=1}^{M} D(0,t_i) \left(\frac{V_j(t_{i-1})^+ + V_j(t_i)^+}{2} \right) B$$

where $B = \mathbb{Q}(\omega = \omega_j | \tau = t_i)(F_\tau(t_i) - F_\tau(t_{i-1}))$

To compute these probabilities conditional to default, Rosen and Saunders proposed to model the credit risk, the market risk and the joint market-credit co-dependence as following :

3.2.1. Credit risk model

At time t we define the counterparty creditworthiness indicator as :

$$Y_c = \Phi^{-1}(F_\tau(t)) \rightarrow N(0,1)$$

If we denote by Z a systematic factor (standard normal random variable), and by ρ_c the correlation coefficient between Y_c and Z, we can write :

$$Y_c = \rho_c Z + \sqrt{1 - \rho_c^2} \varepsilon_c \qquad (2)$$

$\varepsilon_c \rightarrow N(0, 1)$ independent of Z.

3.2.1. Market risk model

It is built directly from simulated exposures (by Monte Carlo over N time points and M scenarios). As in credit risk model, Rosen and Saunders used a standard normal random variable X to describe exposures. At each time point, The value of X should correspond to realized exposure scenario :

$$\omega = \omega_j \Leftrightarrow C_{j-1} \leq X \leq C_j; j = 1, 2, ..., M$$

$(C_j)_{0 \leq j \leq M}$ must match the scenario probabilities, that's why Rosen and Saunders proposed the following expression of C_j :

$$C_j = \begin{cases} -\infty & j = 0 \\ \psi^{-1} & j = 1, 2, ..., M-1 \\ +\infty & j = M \end{cases}$$

Where :

$$Q_j = \sum_{k=1}^{j} q_k; j = 1, 2, ..., M$$

q_k denotes the probability of the scenario k. We suppose all scenarios are equiprobable : $q_k = 1/M$

If ρ_m is the correlation coefficient of X and Z, we write :

$$X = \rho_m . Z + \sqrt{1 - \rho_m^2} . \epsilon_m \qquad (3)$$

$\varepsilon_m \rightarrow N(0,1)$ independent of Z and ε_c

3.2.3. Joint market-credit co-dependence

Using equations (2) and (3), we calculate the correlation of X and Y_c as following:

$$\begin{cases} Y_c = & \rho_c . Z + \sqrt{1 - \rho_c^2} . \epsilon_c \\ X = & \rho_m . Z + \sqrt{1 - \rho_m^2} . \epsilon_m \end{cases}$$

$$\Rightarrow Cov(X, Y_c) = E[XY_c] = \rho_m \rho_c$$

Then we can link X and Y_c with the same above philosophy:

$$X = \rho . Y_c + \sqrt{1 - \rho^2} . \epsilon \qquad (4)$$

Where : $\rho = \rho_m \rho_c$ and $\varepsilon \rightarrow N(0,1)$ independent of Y_c.

3.2.4. Conditional probabilities calculation

The final step is to explicitly compute probabilities conditional to default :

• For the alternative discretization:

$$
\begin{aligned}
&\mathbb{Q}(\omega = \omega_j | \tau = t_i) \\
&= \mathbb{Q}\left(C_{j-1}(t_i) \leq X \leq C_j(t_i) | Y_c = \psi^{-1}(F_\tau(t_i))\right) \\
&= \mathbb{Q}\left(C_{j-1}(t_i) \leq \rho.Y_c + \sqrt{1-\rho^2} \leq C_j(t_i) | Y_c = \phi^{-1}(F_\tau(t_i))\right) \\
&= \mathbb{Q}\left(C_{j-1}(t_i) \leq \rho\psi^{-1}(F_\tau(t_i)) + \sqrt{1-\rho^2.\epsilon} \leq C_j(t_i)\right) \\
&= \mathbb{Q}\left(\frac{C_{j-1}(t_i) - \rho\psi^{-1}(F_\tau(t_i))}{\sqrt{1-\rho^2}} \leq \epsilon \leq \frac{C_j(t_i) - \rho\psi^{-1}(F_\tau(t_i))}{\sqrt{1-\rho^2}}\right) \\
&= \psi\left(\frac{C_j(t_i) - \rho\psi^{-1}(F_\tau(t_i))}{\sqrt{1-\rho^2}}\right) - \psi\left(\frac{C_{j-1}(t_i) - \rho\psi^{-1}(F_\tau(t_i))}{\sqrt{1-\rho^2}}\right)
\end{aligned}
$$

• Using bivariate normal distribution:

$$
\begin{aligned}
&\mathbb{Q}(\omega = \omega_j | \leq t_{i-1} \leq \tau \leq t_i) \\
&= \mathbb{Q}\left(C_{j-1}(t_i) \leq X \leq C_j(t_i) | t_{i-1} \leq \tau \leq t_i\right) \\
&= \psi_2\left(C_j(t_i), \psi^{-1}(F_\tau(t_i)), \rho\right) - \psi_2\left(C_{j-1}(t_i), \psi^{-1}(F_\tau(t_i)), \rho\right) \\
&- \psi_2\left(C_j(t_i), \psi^{-1}(F_\tau(t_{i-1})), \rho\right) + \psi_2\left(C_{j-1}(t_i), \psi^{-1}(F_\tau(t_{i-1})), \rho\right)
\end{aligned}
$$

Where ψ_2 denotes the cumulative bivariate normal distribution function :

$$
\psi_2(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} exp\left(-\frac{u^2 + v^2 - 2\rho uv}{2(1-\rho^2)}\right) dudv
$$

3.3. IPFP algorithm

3.3.1. Definition

IPFP (Iterative Proportional Fitting Procedure) algorithm is a procedure for adjusting a table of data cells such that they add up to selected totals for both columns and rows of the two-dimensional table.

Let's consider the following example :

Selected Marginal (Column) Totals

Selected Marginal (Row) Totals	11	9	8
5	1	2	1
15	3	5	5
8	6	2	2

Seed Cells

The objective of the IPFP consists in an adjustment of the matrix "Seed Cells" *S:= (1,2,1);(3,5,5);(6,2,2)* to obtain the Fitted matrix *F:=(a,b,c);(d,e,f);(g,h,i)* which must verify :

$$
\begin{cases}
a+b+c &= 5 \\
d+e+f &= 15 \\
g+h+i &= 8 \\
a+d+g &= 11 \\
b+e+h &= 9 \\
c+f+i &= 8
\end{cases}
$$

The algorithm consists in the following steps :

• Step 1 : Each row of seed cells was proportionally adjusted to equal the marginal row totals (each cell was divided by the actual sum of the row of cells, then multiplied by the marginal row total) :

$$
\begin{cases}
S(1,1) & \leftarrow \frac{S(1,1)}{S(1,1)+S(1,2)+S(1,3)} * 5 \\
S(1,2) & \leftarrow \frac{S(1,2)}{S(1,1)+S(1,2)+S(1,3)} * 5 \\
S(1,3) & \leftarrow \frac{S(1,3)}{S(1,1)+S(1,2)+S(1,3)} * 5 \\
S(2,1) & \leftarrow \frac{S(2,1)}{S(2,1)+S(2,2)+S(2,3)} * 15 \\
S(2,2) & \leftarrow \frac{S(2,2)}{S(2,1)+S(2,2)+S(2,3)} * 15 \\
S(2,3) & \leftarrow \frac{S(2,3)}{S(2,1)+S(2,2)+S(2,3)} * 15 \\
 & \ldots
\end{cases}
$$

• Step 2 : Each column of (already row-adjusted) cells was proportionally adjusted to equal the marginal column totals :

$$
\begin{cases}
S(1,1) & \leftarrow \frac{S(1,1)}{S(1,1)+S(2,1)+S(3,1)} * 11 \\
S(1,2) & \leftarrow \frac{S(2,1)}{S(1,1)+S(2,1)+S(3,1)} * 11 \\
S(1,3) & \leftarrow \frac{S(3,1)}{S(1,1)+S(2,1)+S(3,1)} * 11 \\
S(2,1) & \leftarrow \frac{S(1,2)}{S(1,2)+S(2,2)+S(3,2)} * 9 \\
S(2,2) & \leftarrow \frac{S(2,2)}{S(2,1)+S(2,2)+S(2,3)} * 9 \\
S(2,3) & \leftarrow \frac{S(3,2)}{S(2,1)+S(2,2)+S(2,3)} * 9 \\
 & \ldots
\end{cases}
$$

• Step 3 : The above steps were repeated until the selected level of convergence was reached.

To be sure that the algorithm converges, these points should be verified :
– Values in the Seed matrix must be non-zero.
– The sum of marginal column totals is equal to the sum of marginal row totals :

$$11 + 9 + 8 = 5 + 15 + 8$$

In this case, two or three iterations (iteration = Step 1 + Step 2) are sufficient for the algorithm convergence for a given confidence level.
With a simple C++ program, these are the results which show the convergence for different number of iterations:

	11	9	8
5	1	2	1
15	3	5	5
8	6	2	2

N=3 →

	10,927	9,044	8,014
4,957	1,722	2,176	1,058
14,907	4,468	5,226	5,213
8,122	4,737	1,642	1,743

	11	9	8
5	1	2	1
15	3	5	5
8	6	2	2

N=5 →

	10,989	9,007	8,002
4,994	1,766	2,170	1,058
14,986	4,556	5,215	5,215
8,018	4,668	1,622	1,729

	11	9	8
5	1	2	1
15	3	5	5
8	6	2	2

N=15 →

	11,000	9,000	8,000
5,000	1,773	2,169	1,058
15,000	4,572	5,213	5,216
8,000	4,655	1,619	1,726

N : number of iterations

3.3.2. Application to WWR

In our situation, we would like to compute probabilities conditional to default: $Q(\omega = \omega_j \mid t_{i1} \leq \tau \leq t_i)$ using this algorithm. Our Seed matrix is $P(i,j)_{i=1..N, j=1..M}$ such as:

$$P(i,j) = \mathbb{Q}(\omega = \omega_j | t_{i1} \leq \tau \leq t_i);\ i = 1..N, j = 1..M$$

And :

$$\sum_{j=1}^{M} P(i,j) = F_\tau(t_i) - F_\tau(t_{i-1}), \forall i \in [1, 2, ..., N]$$

$$\sum_{j=1}^{M} P(i,j) = \frac{1}{M}, \forall j \in [1, 2, ..., M]$$

	$1/M$	$1/M$	.	.	.	$1/M$
$F_\tau(t_1)-F_\tau(t_0)$	Q_{11}	Q_{12}	.	.	.	Q_{1M}
$F_\tau(t_2)-F_\tau(t_1)$	Q_{21}	Q_{22}	.	.	.	Q_{2M}
.	.	.				.
.	.	.				.
.	.	.				.
$F_\tau(t_N)-F_\tau(t_{N-1})$	Q_{N1}	Q_{N2}				Q_{NM}

The margin column vector is standardized : $\sum_{j=1}^{M} \frac{1}{M} = 1$.

The margin row vector should also be normalized :

$$F_\tau(t_i) - F_\tau(t_{i-1}) \leftarrow \frac{F_\tau(t_i) - F_\tau(t_{i-1})}{F_\tau(t_N) - F_\tau(t_0)}, \forall i \in [1, 2, ..., N]$$

The question now is how to initialize our seed matrix P so that after applying IPFP, we obtain :

$$P(i,j) \simeq \mathbb{Q}\left(\omega = \omega_j | t_{i1} \leq \tau \leq t_i\right); i = 1..N$$

One proposition is described as following :

- First, we normalize expositions : $V_j(t_i)^+ = \frac{V_j(t_i)^+}{max_{i,j}(V_j(t_i)^+)}$

- Second, we introduce $\theta = \frac{\rho}{1-\rho^2}$ ρ = $correlation(Exposure,credit)$ exactly as in equation 4

- Third, we initialise our matrix P as : $P(i,j) = exp(\theta.V_j(t_i)^+)$

- Finally, we normalize P ($P(i, j)$ are probabilities):

$$P(i,j) = \frac{P(i,j)}{\sum_{i,j} P(i,j)}$$

4. CONCLUSION

Through this paper, the pricing of CVA with or without WWR has been presented. The Rosen and Saunders method has been detailed. One can use the alternative descretization or the bivariate normal distribution to calculate the probabilities conditional to default. However, we didn't study how we can calibrate and calculate the correlation between exposures and counterparty credit quality, this parameter is an input of the model supposed known.
We also presented the IPFP algorithm which is powerful and allows very fast calculation of the probabilities without using the Gaussian copula or Gaussian distribution and its inverse.

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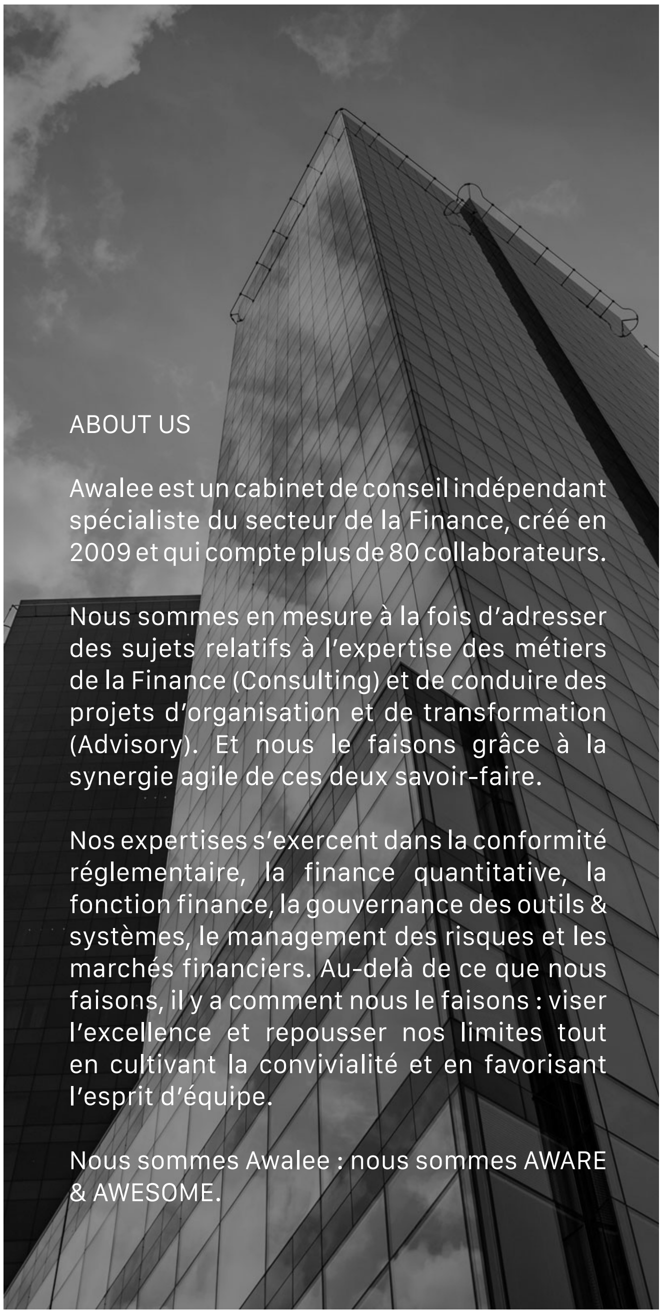

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