AWALEE NOTES

Interest rate modeling with shadow short rate model in negative interest rate environment

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Summary: In this note, we consider a variant of the shadow short rate model in the context of negative rate. We discuss how to use this model to price interest rate derivatives such as caplet/floorlet and swaption.

All model parameters and strikes of IR derivatives that we use for the simulation in this note are chosen arbitrarily. We do not use any market data from any source to carry out the simulation.

1. INTRODUCTION

Negative interest rates are recently adopted in many major financial markets for various reasons. This is a signal that traditional policy options are no longer effective and new extreme measures need to be explored. European Central Bank (ECB) was the first major institution of its kind to cut one of its benchmark rates below zero in 2014. Sweden's Riksbank, the world's oldest central bank, applied the Negative Interest Rate Policy (NIRP) in February, 2015. At the beginning of 2016, The Bank of Japan pushed interest rates below zero after years of adopting the Zero Interest Rate Policy (ZIRP) i.e. keeping the rates extremely low but still positive. These institutions have kept their rates negative from since to stimulate the economy and get inflation up. Indeed, a weaker Swedish krona would make goods coming into the country more expensive, hence, raising domestic prices. The same goes for a weak euro or yen. However, countries like Denmark and Swiss, by applying NIRP, are aiming to reverse the intense appreciation pressures on their own currencies. They want to keep their currencies steady against Euro to protect trade with the euro zone.

One of the important features of Negative Interest Rate Environment (NIRE) is the stickiness property i.e. rates are persistently close to 0 and the corresponding volatilities are small during a long period of time. Moreover, we observe that, in negative IR environment, the distribution of future short rates is highly asymmetric. Besides, the negativity of interest rates like Libor rates leads to a change in terms of volatility quote. For Caplet/Floorlet (CF), we use now the normal model like Bachelier's instead of the Black-Scholes model to compute the implied volatility.

In this note, from the point of view of IR-derivative pricing, we consider a short rate model which is originally used in the economies where ZIRP is applied such as in American economy. In such economies, short-term rates are positive but persistently small during a long period of time. We consider a slight modification for this model by imposing a possibly negative floor on the short rate and we would like to see how we can price CF and Swaption with this model in NIRE when the strikes of CF and Swaption are very low or even negative. Such a situation is typical for IR modeling in the negative rate context.

In the present note, we will restrict ourselves to a mono-curve setting in which all rate (ZC rate, Libor rate, etc) are computed by using the scenarios of (instantaneous) short rate.

2. SHADOW SHORT RATE MODEL

Shadow Short Rate (SSR) Models were introduced originally to model IRs in Zero Lower Bound (ZLB) context i.e. IRs are positive but small. The fundamental idea for shadow models is to use ZLB short rates

$$\bar{r}_t = max(r_t, 0)$$

where r_t is a Vasicek short rate

$$dr_t = \lambda(\theta - r_t)d_t + \sigma dW_t$$

with $\lambda > 0$ and W_t is a standard Brownian Motion.

We observe that $P(\bar{r}_t = 0) = P(r_t < 0) > 0$ and $P(\bar{r}_t < 0) = 0$ for all t. Hence, the density of the short rate $\bar{r}_t$ is discontinuous at 0. This is an important feature of the model for generating rates persistently close to 0 for a certain period of time.

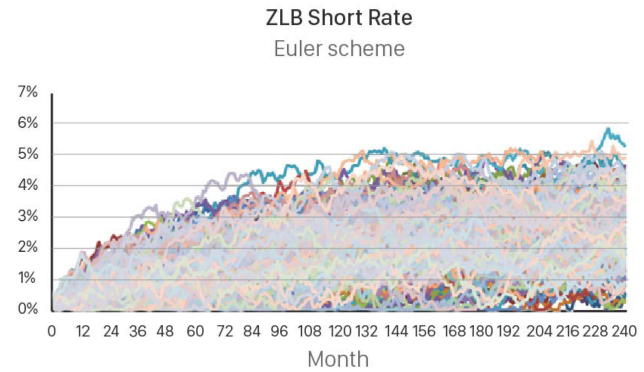
This model can be adapted to the negative rate context in a quite brutal way by adjusting the floor (lower bound) in the definition of the short rate model

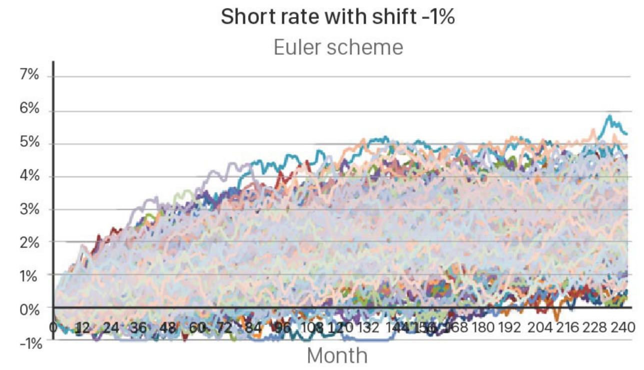
$$\bar{r}_t = max(r_t, -s)$$

where s is a positive constant. We can even choose s to be a time-dependent function. Note that we do not add an additive term to our short rate model to fit the initial ZC curve since this term change the lower bound of the short rate. We would like to explicitly fix the floor that way to see the impact of the shift on the pricing procedure. Obviously, a drawback of this method is that, when we impose a floor $-s < 0$, the probability distribution of $\bar{r}_t$ has a mass at $-s$ instead of 0, hence, the stickiness property around the level 0 might be violated.

We plot here the scenarios of monthly short rates. Note that, since a Vasicek short rate has a closed-form representation, an exact simulation scheme, which conserves the law of r_t, is available. However, in terms of pricing IR derivatives, an Euler scheme is already good enough. Throughout this chapter, without other indications, we use the following set of model parameters

r_0	lambda	theta	sigma	shift	Discretization step *dt*
0.05%	10.00%	3.00%	0.50%	-1%	0.08333





3. ZC BOND PRICING

In a shadow rate model, a closed form formula for ZC price is not available. Hence, a numerical approach for pricing ZC bond is needed. In the present note, we use the MC method to compute ZC bond price. Assume that we would like to compute the ZC price $P(0,T)$. We use the Euler scheme to generate N scenarios of short rate $\bar{r}_t$ from 0 to T with discretization step $dt = 1/12$ i.e. we generate the monthly short rate. Precisely, we generate a matrix $N \times (12T+1)$ of short rate $\bar{r}_{ij} = max\ (r_{ij}, -s)$ given by the Euler's scheme

$$r_{i,j+1} = r_{i,j} + \lambda(\theta - r_{i,j}) + \sigma\sqrt{dt} \times \epsilon_{i,j+1}$$

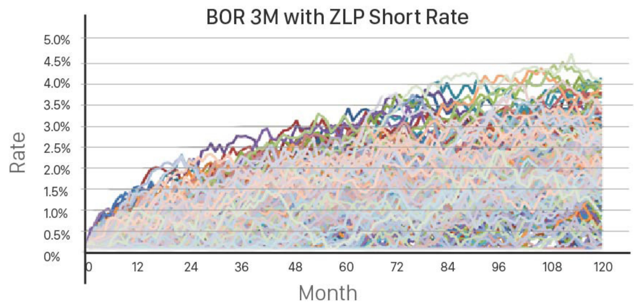
where $(\epsilon_{i,j})$ is a matrix of i.i.d. standard Gaussians. Note that $r_{i,j}$ is the value of short rate of the step *jth* in the *ith* scenario.

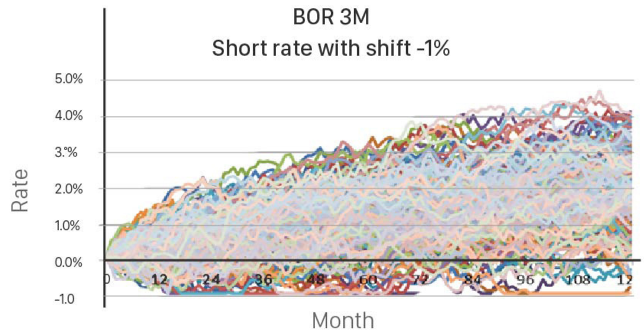
Next, we apply the pricing formula to approximate the ZC price.

From now on every pricing formula in our note will be written under the risk neutral probability:

$$\begin{aligned} P(0,T) &= E\left[\exp\left(-\int_0^T \bar{r}(s)ds\right)\right] \\ &\approx \frac{1}{N}\sum_{i=1}^{N}\exp\left(-\int_0^T \bar{r}_i(s)ds\right) \\ &\approx \frac{1}{N}\sum_{i=1}^{N}\exp\left(-dt\sum_{j=1}^{12T+1}\bar{r}_{ij}\right) \end{aligned}$$

In order to compute the value of a bond at an arbitrary instance t, we use $\bar{r}_t$ as the initial value and generate N scenarios of short rate in the period $[\ t\ ,\ t+T\ ]$.





Note that, in an affine term structure model as Vasicek one, given a maturity T, the ZC price $P(t,T)$ at t is a function in r_t, the level of short rate at t. The function we implement here has the same prototype, it takes r_t as the initial value to generate scenarios of short rate and compute the ZC price.

In order to accelerate a little bit the convergence rate of the MC method, a variance reduction technique such as the antithetic variates method can be applied. We generate a matrix g with N scenarios of Gaussians and create $2N$ scenarios of short rates where the N first scenarios are computed by using g and the last N scenarios are computed from $-g$.

More efficiently, we can use quasi MC method with low discrepancy sequences such as Sobol one. We start off by generating a matrix of Sobol numbers. From each Sobol number we generate one standard Gaussian by using either a Box-Muller procedure or the inverse normal distribution function. From these Gaussians we create once again the matrix of short rate and numerically compute the ZC bond prices as mentioned above.

4. CF PRICING

In a Vasicek short rate model, Caplet and Floorlet (CF) prices can be analytically computed. However, when we impose a floor on the Vasicek short rate, a closed-form formula is no longer available. We will use again the MC method to compute CF prices.

Consider a caplet with maturity T and the pay-off $\delta(L(T,T+\delta) - K)^+$ *at* $T+\delta$ where $L(T,T+\delta)$ is the Libor rate with maturity T and the payment date $T+\delta$.
For the sake of simplicity, we compute the caplet price at t = 0 with the short rate
$\bar{r}_t = max\ (r_t, -s)$ where s ≥ 0.

$$\pi_0^{\text{caplet}} = \delta E\left[e^{-\int_0^{T+\delta}\bar{r}(s)ds}(L(T,T+\delta)-K)^+\right]$$

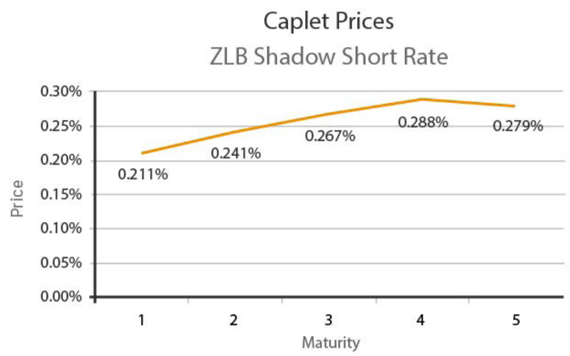
$$\approx \frac{1}{N}\sum_{i=1}^{N}\delta e^{-\int_0^{T+\delta}\bar{r}_1(s)ds}(L_i(T,T+\delta)-K)^+$$

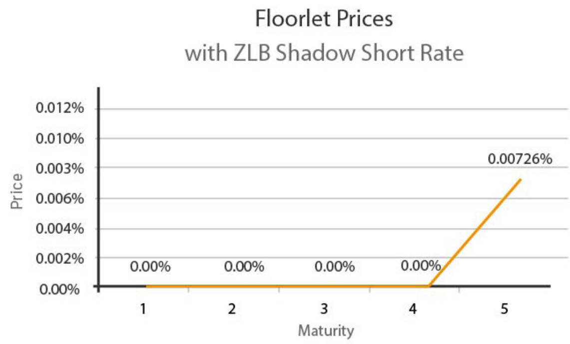
To price a caplet, we simulate N scenarios of monthly short rate over the period of time $[0,T+\delta]$. First of all, this matrix of scenarios will be used to compute $\int_0^{T+\delta}\bar{r}_i(s)ds$ in the pricing formula. Next, for i-th scenario with $i=1,2...N$, we have to compute the Euribor $L_i(T,T+\delta)$ based on the value $r_{i,12T+1}$ of short rate at T in the scenario i. Note that

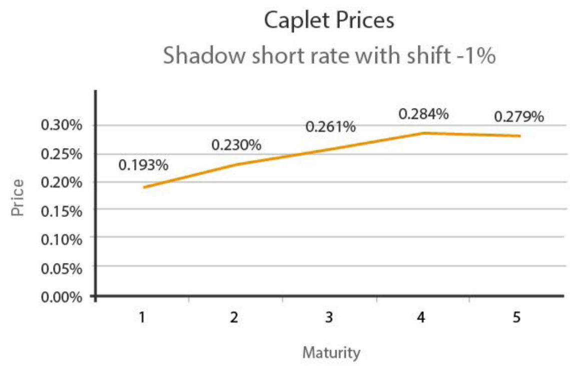
$$L_i(T,T+\delta) = \frac{1}{\delta}\left(\frac{1}{P_i(T,T+\delta)} - 1\right)$$

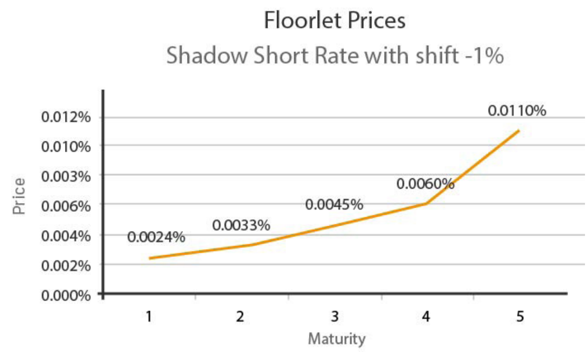
Hence, for each scenario i, we must generate whole new scenarios of short rate on $[T,T+\delta]$ to compute the ZC price $P_i(T,T+\delta)$ by using $r_{i,12T+1}$ as the initial value of the short rate. In other words, *we carry out a nested Monte Carlo procedure*. This mechanism is quite expensive in terms of execution time.

We implement functions to compute CF prices w.r.t. 2 shifts 0 and -1% with *low or even negative strikes*. Recall that we use the same set of model parameters as above. An important remark is that, for ZLB short rate ($s=0$), we often observe the price 0 for floorlets with short maturities since too many scenarios of Libor rates are above the level of corresponding strikes. With the shift -1%, the model prices that we obtain are more "reasonable".









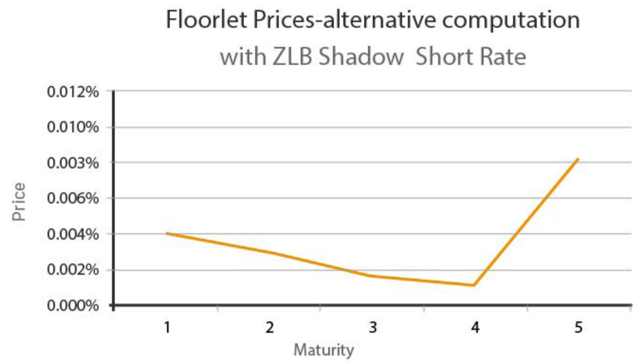
In the ZLB case, to improve the downside of Monte Carlo method when the floorlet is OTM, we can use the Call Put Parity for CF. Since

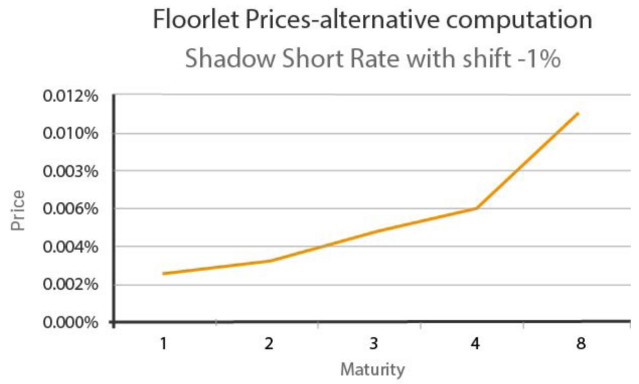
$$\delta(L(T,T+\delta)-K)^+ - \delta(K-L(T,T+\delta))^+$$
$$= \delta(L(T,T+\delta)-K)$$

we have

$$\pi_0^{\text{caplet}} - \pi_0^{\text{floorlet}} = \pi_0^{\text{swap-payer}}(T, T+\delta)$$
$$= P(0,T) = (1+\delta K)P(0, T+\delta)$$

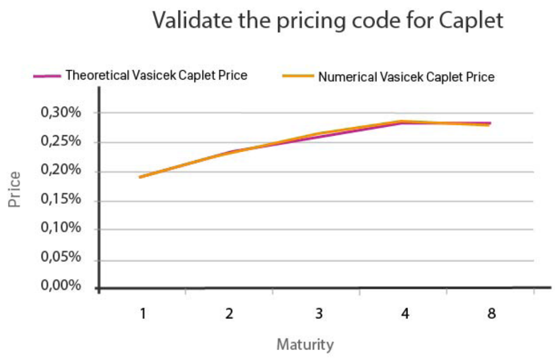
Hence, the price of a floorlet can be computed via that of a caplet and ZC bonds of maturity T and $T+\delta$.

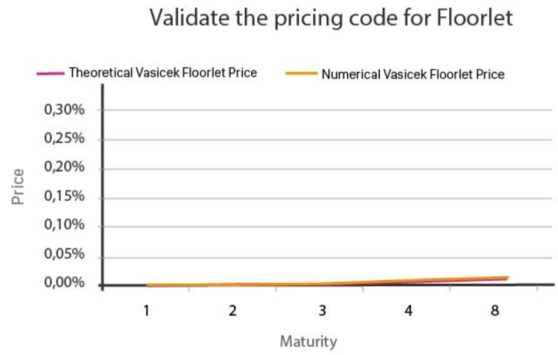




We observe that, this computation method changes completely the shape of the floorlet curve in ZLB shadow short rate model. However, when we impose a floor -1%, the floorlet curve shape barely changes and the floorlet price is increasing in maturity. This is again an evidence to show that pricing with negative shift is better in our example.

Recall that, in a Vasicek model, we can compute the CF prices analytically (c.f. (D.Brigo and F.Mercurio, 2007)). Hence, in order to verify our pricing code for CF, we can modify the code to get Vasicek short rate. Then, our code gives numerical results for CF prices. We compare the results with those computed via closed-form formulae.





5. SWAPTION PRICING

Consider of swaption of maturity T associated to a swap with tenor $\{T_0, \ldots, T_n\}$. For the sake of simplicity, we assume that $T = T_0$ and we determine the swaption price at $t = 0$.

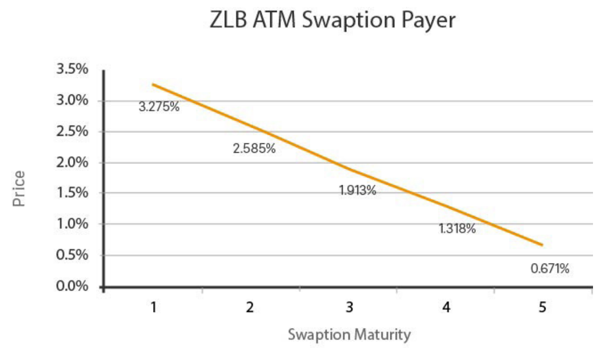
$$\pi_0^{\text{swaption}} = E\left[e^{-\int_0^T \bar{r}(s)ds}(\pi_T^{\text{swap}})^+\right]$$
$$\approx \frac{1}{N}\sum_{i=1}^{N} e^{-\int_0^T \bar{r}_1(s)ds}(\pi_{T,i}^{\text{swap}})^+$$

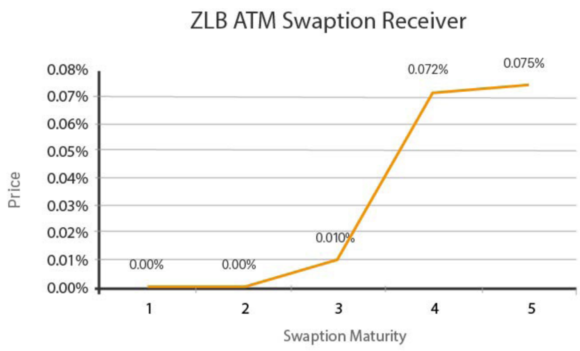
Recall that the swap payer price with the tenor $\{T_0, \ldots, T_n\}$ is given by

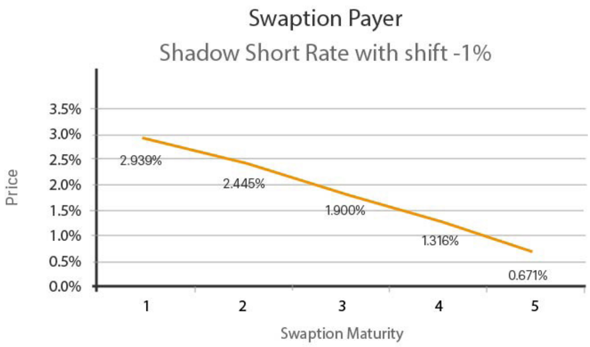
$$\pi_t^{\text{swap-payer}} = P(t, T_0) - P(t, T_n) - K\sum_{i=1}^{n}\delta_i P(t, T_i)$$

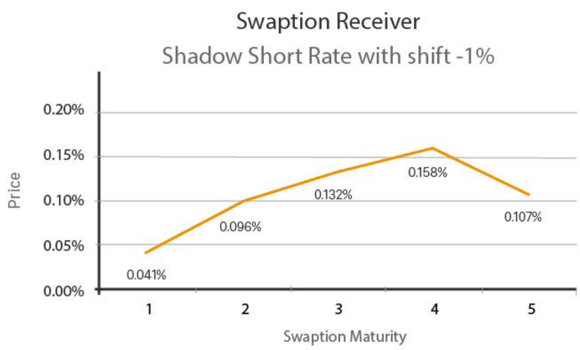
and $\pi_t^{\text{swap\_receiver}} = -\pi_t^{\text{swap\_payer}}$ where $\delta_i = T_i - T_{i-1}$.

Like CF Pricing case, we need to invoke a nested MC procedure. In Swaption case, we numerically compute the swap price at T for each scenario i. It is easy to see that computing a swap price with long maturity will be much more time consuming than computing a Libor $L(T, T+\delta)$ as in CF case. Hence, the execution time for swaption pricing is considerably longer than that of CF pricing.
Besides, we observe the same phenomenon as in the CF case when we price At-the-money (ATM) Swaption Payer and Receiver with ZLB shadow rate and the shadow rate with negative shift -1%. For ZLB shadow rate, many short-term swaption receiver prices are equal to 0 since too many scenarios of the corresponding swap receiver prices go negative.









a. A few words on how to validate the Swaption Pricing code

i. Closed-from formula for PRS Pricing in Vasicek model

Consider a swaption payer with maturity T, strike K and the tenor of the associated swap $\{T_0, T_1, \ldots, T_n\}$ where $T_0 = T$.
Then, the pay-off at T of this swaption payer is

$$H_T = (1 - \sum_{i=1}^{n} c_i P(T, T_i))^+$$

where $c_i = (T_i - T_{i-1}) * K$ for all $i = 1, \ldots, n - 1$ and $c_n = 1 + (T_n - T_{n-1}) * K$.

Next, we apply the following Jadamshian's decomposition to transform H_T into a sum of pay-offs of certain puts.

Step 1 : Find the unique r^* such that

$$\sum_{i=1}^{n} c_i P(r^*, T, T_i) = 1$$

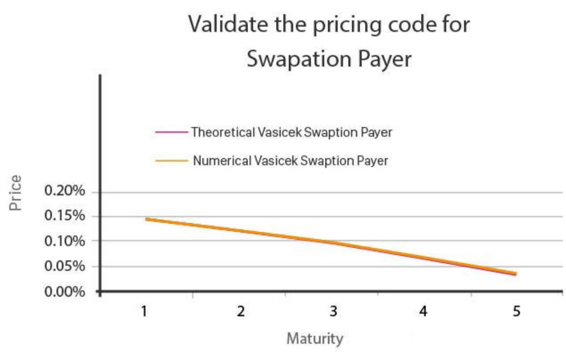
Practically, we have to use a numerical method (Newton's method for instance) to compute r^*.

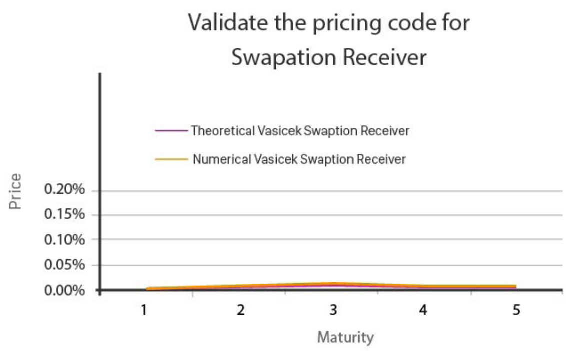
Step 2 : Put $K_i = P(r^*, T, T_i)$. Since the ZC price is a decreasing function in short rate, we can rewrite the pay-off H_T as follows

$$H_T = \sum_{i=1}^{n} c_i (K_i) - P(T, T_i))^+$$

On the other hand, we can compute explicitly the price of a put on a ZC bond in Vasicek model (c.f. (D.Brigo and F.Mercurio, 2007)). Hence, we can compute explicitly the swaption payer in Vasicek model. The Swaption receiver price can be computed in the same way by using the closed-form formula for a call on a ZC bond.

To justify our PRS pricing code, we modify our code to get the Vasicek short rate and numerically compute the swaption prices. Next, we implement functions which use the closed-form formulae in Vasicek model to price swaptions. We compare these two results. We can see that the theoretical and the numerical results are very close to each other.





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