AWALEE NOTES

The Application of Kalman Filter in the Stochastic Model Estimation of Commodities

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1. INTRODUCTION

The Kalman Filter, firstly proposed by R.E. Kalman (1960, cf. [1]), is an algorithm to estimate the optimal parameters of a linear dynamical system. Precisely, the idea of Kalman filtering is to predict a state vector's mean and variance after updating all the information from the previous period. Two equations need to be specified: the **transition equation** and the **measurement equation**. By combining these two equations, Kalman Filter minimizes the covariance in the updated estimator.

The transition equation, which is a linear dynamical system, assumes that the state at time t is based on the state at time $t-1$. It is computed from the mean and variance of the state vectors from the previous period.

The transition equation is driven by the stochastic process of innovations:

$$x_t = a + Ax_{t-1} + \varepsilon_t, \quad t = 1, 2, \ldots T \tag{1}$$

The measurement equation describes the relation between the observed variable and the state variables:

$$y_t = b + B'x_t + \epsilon_t, \quad t = 1, 2, \ldots T \tag{2}$$

With:

x_t a vector of state variables
y_t a vector of observed variables
a and A are respectively a vector and a matrix of parameters in the transition equation
b and B' are also a vector or a matrix of parameters in the measurement equation ε_t (state noise) and ϵ_t (measurement noise) are both vectors of disturbances

Indeed, the state variable x_t is estimated by an update of the previous value x_{t-1}. The observed variable y_t tries to measure correctly the unobserved variable x_t by the measurement equation.

The Kalman filter assumes that the disturbances ε_t and ϵ_t are normally distributed and serially uncorrelated, with null Expected Value and Variance W for ε_t and V [1] for ϵ_t.

$$\varepsilon_t \sim N(0, W)$$

$$\epsilon_t \sim N(0, V)$$

[1] The variance V could be assumed to be diagonal.

Moreover, ε_t and ϵ_t are independent and uncorrelated with the initial state at all time periods.

The process of Kalman Filter can be summarised in the following steps. Before starting the algorithm, more variables should be clarified.

The expectation of x_t, which is $\bar{x}_{t|s}=(x_t \mid y_0, \ldots y_s)$, shows that the variable x at time t is estimated based on the measurement value y of all the previous time 0 to s.

The covariance of the error of the estimate $\bar{x}_{t|s}$ is given by:

$$\sum_{t|s} = E[(x_t - \bar{x}_{t|s})(x_t - \bar{x}_{t|s})^T] \tag{3}$$

By applying the transition equation (time update):

$$x_{t+1}|y_t = \alpha + Ax_t|y_t + \varepsilon_t \tag{4}$$

By applying the transition equation (time update):

$$\bar{x}_{t+1|t} = \alpha + A\bar{x}_{t|t} \tag{5}$$

And the variance:

$$\sum_{t+1|t} = E\left[\left(A\bar{x}_{t|t} - Ax_t - \varepsilon_t\right)\left(A\bar{x}_{t|t} - Ax_t - \varepsilon_t\right)^T\right] \tag{6}$$

$$= A\sum_{t|t} A^T + W$$

By applying the measurement equation, the Kalman filter shows that:

$$\bar{x}_{t|t} = \bar{x}_{t|t-1} + \sum_{t|t-1} B^T \left(B \sum_{t|t-1} B^T + V\right)^{-1} (y_t - B\bar{x}_{t|t-1}) \tag{7}$$

$$\sum_{t|t} = \sum_{t|t-1} - \sum_{t|t-1} B^T \left(B \sum_{t|t-1} B^T + V\right)^{-1} B \sum_{t|t-1} \tag{8}$$

As a conclusion, by starting with initial values x_0 and Σ_0, according to the equation (1), we get $\bar{x}_{0|0}$ and $\Sigma_{0|0}$.

By applying the equation (6), we have $\bar{x}_{1|0}=A\bar{x}_{0|0}$ and $\Sigma_{1|0} = A\Sigma_{0|0} A^T + W$.

The process is repeated until the end of observation y.

2. THE BENEFITS OF KALMAN FILTER IN COMMODITY MARKET MODELS

2.1. Theory of Storage and Factor Models in Commodity Market

This section will explain how we can estimate the parameters of factor models for commodities by using Kalman Filter. According to the theory of storage, several variables are not observed, which conducts to the possible inaccuracy estimation of stochastic models in commodity markets.

The theory of storage explains the relationship between the spot and futures prices in commodity market. The spread between the spot and futures prices is related to the charge of storing the commodity. The spread is composed of capital charge, storage costs and the "price of storage" (H. Working, 1949, cf. [2]), which increases with the level of inventories since high inventories imply scarce residual storing capacities.

When inventories are depleted, the price of storage turns negative as free storage space is abundant and convenience yield (the benefit) becomes positive.

"Backwardation" corresponds to a situation where the futures term structure curve is downward sloping. According to the theory of storage, backwardation occurs when the convenience yield is greater than the cost of storage (inventory is low). The opposite situation is the "Contango".

Based on the theory of storage and the model developed by Schwartz (1997, cf. [3]), a two-factor model has been proposed and tested[2] for copper, oil and gold. These two-factor are the spot price S of the commodity and the instantaneous convenience yield δ. The instantaneous convenience yield is not an intuitive concept and thus no market data of convenience yield exists. Even for the spot price, in some commodities, the spot prices data do not exist. And the nearest futures prices are used as the proxy of spot prices.

In the Schwartz Two-Factor Model, the two factors S and δ are assumed to follow the stochastic process as:

$$dS = (\mu - \delta)Sdt + \sigma_S S dB_s \tag{9}$$

$$d\delta = \kappa(\alpha - \delta)dt + \sigma_\delta dB_\delta \tag{10}$$

And under risk-neutral probability, the processes above can be written as:

$$dS = (r - \delta)Sdt + \sigma_s S dB_s^* \tag{11}$$

$$d\delta = [\kappa(\alpha - \delta) - \lambda]\, dt + \sigma_\delta dB_\delta^* \tag{12}$$

The increments of Brownian motion are correlated: $dB_s^* dB_\delta^* = \rho dt$.
κ and λ represent the speed of the convenience yield returning to the mean level α and the risk premium.

By transforming the two processes into risk-neutral measure, the solution to the partial differential equation is:

$$lnF(S,\delta,T) = lnS - \delta\frac{1-e^{-\kappa T}}{\kappa} + A(T) \tag{13}$$

With:

$$\begin{aligned} A(T) &= \left(r - a' + 0.5\frac{\sigma_\delta^2}{\kappa} - \frac{\sigma_s\sigma_\delta\rho}{\kappa}\right)T + 0.25\sigma_\delta^2\frac{1-e^{-2\kappa T}}{\kappa^3} \\ &= + \left(a'\kappa + \sigma_s\sigma_\delta\rho - \frac{\sigma_\delta^2}{\kappa}\right)\frac{1-e^{-\kappa T}}{\kappa^2} \end{aligned} \tag{14}$$

And $a' = a - \lambda/\kappa$

2.2. The Application of Kalman Filter in Three Agricultural Commodities

In the stochastic model, continuous variables are assumed. In some cases, the considered variables or the factors do not exist in the market data or are even unobservable. Those variables are mostly considered on a theoretical level, which reveals the difficulty of estimation in this kind of model. As a result, a proxy of market data or a value derived from a theoretical framework will be used. However, the accuracy of the corresponding variable is questionable. Since both the proxy of market data and the value based on the theoretical model are hard to be justified and probably not coherent from a statistical point of view.

This section explains how the Kalman Filter is used to estimate the unobservable factors in the Schwartz two-factor model.

The two-factor model can be transformed into a State-space representation, which shows how to apply the Kalman filter in the estimation of parameters. As it is showed in the above section, the observation variables y_t used in the two-factor model context will be the futures prices for different maturities $lnF(S,\delta,T)$ and the unobservable state variables x_t in the measurement equation will be the spot price and the instantaneous convenience yield [lnS,δ]. By transforming the stochastic model into state-space form, the two-factor model becomes as described below.

Measurement Equation

$$lnF(S,\delta,T_i) = a_t + A_t[lnS_t,\delta_t]' + \varepsilon_t \tag{15}$$

With $a_t = A(T_i)$ and $A_t = \left[1, \frac{1-e^{-\kappa T_i}}{\kappa}\right]$

[2] It should be noted that, in the original paper, Schwartz tested three models: one-factor, two-factor and three-factor. The conclusion is that the two-factor model outperforms the other two models for the commodities copper, oil and gold. In order to illustrate the Kalman Filter in the commodity market, this note tests only the two-factor model.

We choose $i = 1,...,N$, *i.e.* N different maturities of futures contracts. And we assume that $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = H$.

Transition Equation

$$[lnS_t, \delta_t]' = b_t + B_t\,[lnS_{t-1}, \delta_{t-1}]' + \theta_t \qquad (16)$$

According to the stochastic process:

$$b_t = [(\mu - 0.5\sigma_s^2)\Delta_t, \kappa a \Delta_t]' \text{ and } B_t = \begin{pmatrix} 1 & -\Delta t \\ 0 & -\kappa \Delta t \end{pmatrix}$$

We have the assumption that :

$E(\theta_t) = 0$

$$Var(\theta_t) = \begin{pmatrix} \sigma_s^2 \Delta t & \rho\sigma_s\sigma_\delta\Delta_t \\ \rho\sigma_s\sigma_\delta\Delta_t & \sigma_\delta^2\Delta_t \end{pmatrix}$$

The data used to implement this model is the daily futures prices of three agricultural commodities from January 2006 to July 2007: wheat, soybean, and cocoa. It is important to note that these commodities are all storable. Due to the lower market liquidity for longer maturities contracts and the data availability, four contract maturities are chosen for each product. The contracts of 1 month, 3 months, 7 months and 10 months are chosen.

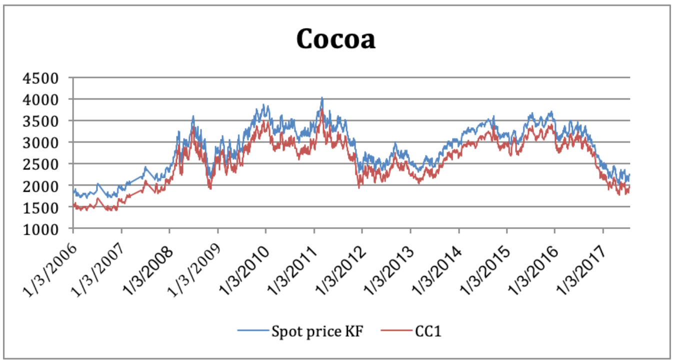
2.2.1 Spot Prices and Convenience Yield

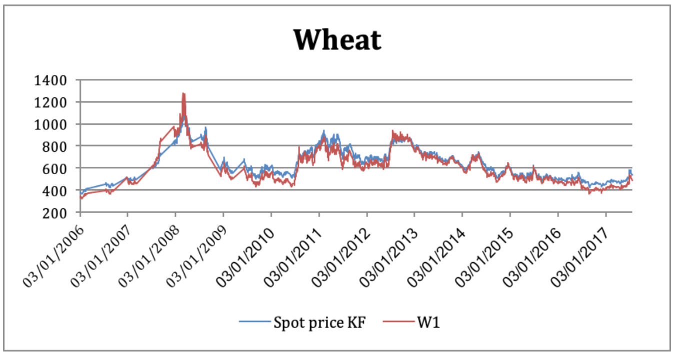
Filtering Spot Prices

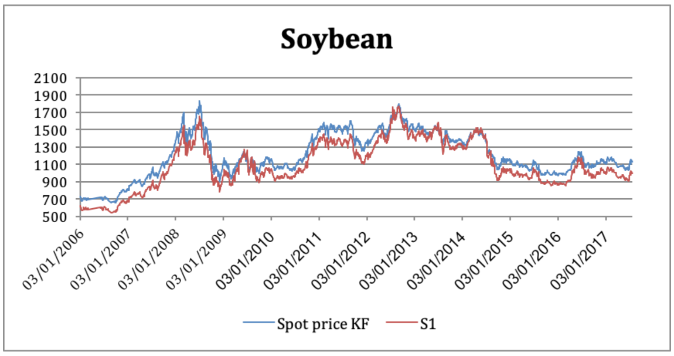
First, we filter the spot prices and convenience yield time series according to the Kalman filter. In the following three graphs, the comparison is conducted between the spot prices by Kalman filter and the nearest futures prices for each product, noting CC1 for futures contracts of cocoa, W1 for futures contracts of wheat and S1 for futures contracts of soybean.

It should be noted that, as in the agricultural physical market, there is a limited liquidity. The data for physical spot prices will be sometimes not available. The price of the nearest futures contract is usually considered as a proxy of spot prices. From the following graphs, it can be seen that the evolution of spot prices of Kalman filter based on the two-factor model is similar to the futures prices. The gap between filtered spot prices and the futures prices could be explained by the fact that even if the chosen futures contracts are close to the maturity, they are still live.

In addition to that, the difference may be due to the limited cost of storage or the expectation of prices (according to the sign of difference, we can distinguish the case of contango or backwardation).







Graph 1: Filtered Spot Prices

Filtering the Convenience Yield

Contrary to the spot price, the convenience yield is not observable in the market (and it does not have a proxy with observable data either). Nevertheless, the convenience yield can be calculated according to the relation between futures prices and spot prices in the theory of storage. However, the cost of storage is difficult to measure. In order to check the results of the filtered convenience yield, we check the relationship between the filtered convenience yield and the spread between the futures prices and the spot prices. The filtered convenience yield is negatively correlated with this spread. The correlation is significant. As it is described in the above section, when the spread is negative, convenience yield is higher than the cost of storage.

	Cocoa	Soybean	Wheat
Correlation with spot prices	-0.592	-0.478	-0.372
p-value	< 2.2e-16	< 2.2e-16	< 2.2e-16

Table 1: Correlation between Spread of Futures and Spot Prices, and Convenience Yield

2.2.2. Estimated Parameters

After obtaining the filtered times series, we apply the maximum likelihood method to estimate the parameters of two-factor models. The estimated parameters are showed in the following table. The two-factor models could be used in the prices of futures and options, which are detailed in Schwartz (1997). The value of risk premium in the three commodities is small and near zero. Cocoa and soybean returns display less mean-reverted features but are more trended than wheat, however wheat prices seem more volatile. All the three commodities prices are highly correlated with the convenience yield, which is linked to the inventory.

	Cocoa	Soybean	Wheat
μ	0.184	0.158	0.030
σ_s	0.325	0.256	0.340
κ	1.08	1.172	0.110
α	0.084	0.043	1.270
σ_δ	0.325	0.236	0.215
ρ	0.55	0.830	0.736
r	0.01	0.030	0.030
λ	-0.015	0.000	0.000
Log Likelihood	-1.013.e6	-9.764.e6	-3 198.e6

Table 2: Estimated Parameters of Two-Factor Model
(All parameters are significant for at least 0.1 level)

3. CONCLUSION AND EXTENDED KALMAN FILTER

The results displayed in this note have showed that Kalman Filter is practical for dealing with the existence of unobservable data and lack of data due to illiquidity in certain markets, among which commodity market often faces this issue. The spot prices and convenience yield obtained with Kalman filter are showed to be consistent with the theory of storage and are considered as an optimal estimator. By using the filtered time series, the parameters estimated in the two-factor models have showed the features in the prices of three commodities under review: cocoa, wheat and soybean.

The aim of this note was to introduce the basic notions of the Kalman Filter and show how it could be useful to model the Commodities Market. A more complex modelling framework would require an **Extended Kalman Filter**. The Extended Kalman Filter is able to deal with the nonlinearity between two time series, which can be very useful to take into account when analyzing financial series.

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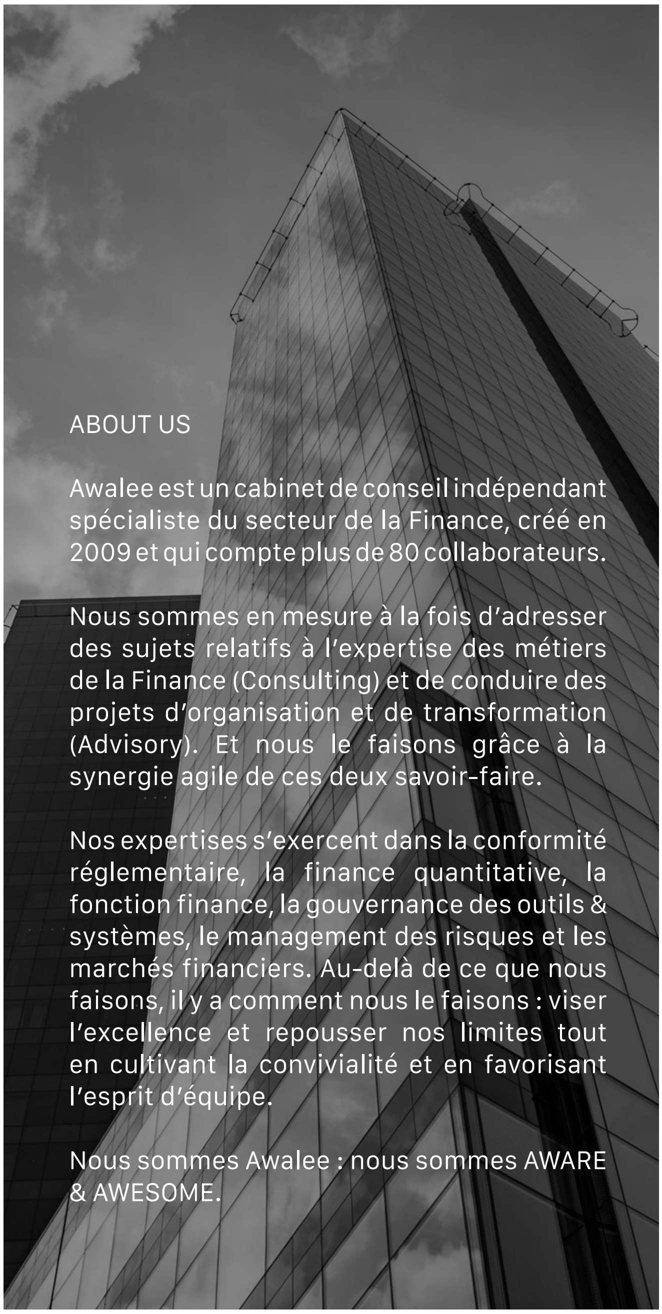

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